1D drift kinetic models with periodic boundary conditions

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1. Introduction

In this report, we propose 1D drift kinetic equations to test the possibility of extracting low order moments from the distribution functions for implicit methods. The model that we present here has periodic boundary conditions, adequate for the closed field line region of the edge. We will address wall boundary conditions for open field lines in the reports for milestones M1.3, M2.4 and M2.5.

2. 1D electrostatic drift kinetics

We consider a plasma with one ion species with charge e and mass m_i, electrons with charge $-e$ and mass m_e, and one species of neutrals with mass

$$m_n = m_i. \tag{2.1}$$

The plasma is magnetized by a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, and we assume that the plasma only varies along magnetic field lines. In this case, the electric field produced by the plasma is electrostatic, $\mathbf{E} = -(\partial\phi/\partial z)\hat{\mathbf{z}}$. The potential $\phi(z, t)$ depends on the position along magnetic field lines z and on time t.

If we assume that the gyroradii are small compared to the length scales of interest, and that the gyrofrequencies are much larger than the frequencies that we want to model (Hazeltine 1973), the distribution functions $f_s(z, v_\parallel, v_\perp, t)$ of the different species $s = i, e, n$ only depend on the component of the velocity parallel to the magnetic field $v_\parallel$ and the magnitude of the velocity perpendicular to the magnetic field $v_\perp$, and are independent of the direction of the velocity perpendicular to the magnetic field. Thus, the distribution functions that in general can depend on three spatial variables $\mathbf{r}$, three components of the velocity $\mathbf{v}$ and the time t depend only on z, $v_\parallel$, $v_\perp$ and t,

$$f_s(\mathbf{r}, \mathbf{v}, t) = f_s(z, v_\parallel, v_\perp, t). \tag{2.2}$$

The equations for the distribution functions of the different species are

$$\frac{\partial f_i}{\partial t} + v_\parallel \frac{\partial f_i}{\partial z} - \frac{e}{m_i}\frac{\partial \phi}{\partial z}\frac{\partial f_i}{\partial v_\parallel} = C_{ii}[f_i] + C_{in}[f_i, f_n], \tag{2.3}$$

$$\frac{\partial f_e}{\partial t} + v_\parallel \frac{\partial f_e}{\partial z} + \frac{e}{m_e}\frac{\partial \phi}{\partial z}\frac{\partial f_e}{\partial v_\parallel} = C_{ee}[f_e] + C_{ei}[f_e, f_i] + C_{en}[f_e, f_n] \tag{2.4}$$

and

$$\frac{\partial f_n}{\partial t} + v_\parallel \frac{\partial f_n}{\partial z} = C_{ni}[f_n, f_i]. \tag{2.5}$$

Here we have included ion-ion and electron-electron collisions, modeled by the Fokker-Planck collision operators $C_{ii}[f_i]$ and $C_{ee}[f_e]$ (Rosenbluth *et al.* 1957), elastic electron-ion and electron-neutral collisions, modeled by the simplified Fokker-Planck collision operator $C_{ei}[f_e, f_i]$ (Braginskii 1958) and the Boltzmann collision operator $C_{en}[f_e, f_n]$, and charge-exchange collisions, represented by the simplified Boltzmann collision operators

$$C_{in}[f_i, f_n] = -\int R_{in}(|\mathbf{v}-\mathbf{v}'|)\,[f_i(\mathbf{v})f_n(\mathbf{v}') - f_i(\mathbf{v}')f_n(\mathbf{v})]\;\mathrm{d}^3v' \tag{2.6}$$

and

$$C_{ni}[f_n, f_i] = -\int R_{in}(|\mathbf{v}-\mathbf{v}'|)\,[f_n(\mathbf{v})f_i(\mathbf{v}') - f_n(\mathbf{v}')f_i(\mathbf{v})]\;\mathrm{d}^3v', \tag{2.7}$$

To simplify our equations, we assume that the function R_{in} is constant (Connor 1977; Hazeltine *et al.* 1992; Catto 1994), finding

$$C_{in}[f_i, f_n] = -R_{in}\,(n_n f_i - n_i f_n) \tag{2.8}$$

and

$$C_{ni}[f_n, f_i] = -R_{in}\,(n_i f_n - n_n f_i)\,, \tag{2.9}$$

where the densities are

$$n_s(z,t) := 2\pi \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \int_0^{\infty} \mathrm{d}v_\perp\, v_\perp f_s(z, v_\parallel, v_\perp, t). \tag{2.10}$$

Note that we can neglect the effect of electron collisions on ions and on neutrals due to the smallness of the electron mass (Braginskii 1958). We have also neglected neutral-neutral collisions because, in current fusion devices, the neutral density is sufficiently small that the neutral-neutral collisions are rare.

The kinetic equations will be solved in the interval $z \in [0, L]$, and we will impose periodic boundary conditions at $z = 0$ and $z = L$,

$$f_s(z = 0, v_\parallel, v_\perp, t) = f_s(z = L, v_\parallel, v_\perp, t). \tag{2.11}$$

Finally, the potential $\phi(z,t)$ is determined by the quasineutrality equation

$$n_i = n_e. \tag{2.12}$$

To solve this equation, we need to treat the equations implicitly as the potential enters only via its effect on $\partial f_i/\partial t$ and $\partial f_e/\partial t$. The need to use implicit methods is one of the reasons why we are trying to extract some of the low order moments from the distribution function, notably the density.

Before we treat the complete problem, we will simplify the treatment of electrons to obtain a system of equations that can be solved with an explicit time advance so that we can compare our implicit schemes with an explicit numerical method. Instead of solving for f_e, we will use a Maxwell-Boltzmann response,

$$n_e(z,t) = N_e \exp\left(\frac{e\phi(z,t)}{T_e}\right), \tag{2.13}$$

where N_e and T_e are constants (see Appendix A for a derivation of the Maxwell-Boltzmann response). Moreover, the full Fokker-Planck ion-ion collision operator $C_{ii}[f_i]$ is a complicated integro-differential operator that we will not implement in the first versions of our drift kinetic code, so we do not include it in the equations for now. Thus, the final simplified model for $f_i(z, v_\parallel, v_\perp, t)$, $f_n(z, v_\parallel, v_\perp, t)$ and $\phi(z,t)$ is given by the equations

$$\frac{\partial f_i}{\partial t} + v_\parallel \frac{\partial f_i}{\partial z} - \frac{e}{m_i}\frac{\partial \phi}{\partial z}\frac{\partial f_i}{\partial v_\parallel} = -R_{in}(n_n f_i - n_i f_n), \tag{2.14}$$

$$\frac{\partial f_n}{\partial t} + v_\parallel \frac{\partial f_n}{\partial z} = -R_{in}(n_i f_n - n_n f_i) \tag{2.15}$$

and

$$n_i = N_e \exp\left(\frac{e\phi}{T_e}\right), \tag{2.16}$$

with periodic boundary conditions (2.11). This system of equations can be solved explicitly because the simple electron model allows one to obtain ϕ as a function of n_i.

3. 1D moment drift kinetics

Instead of solving for $f_s(z, v_\parallel, v_\perp, t)$, we solve for

$$F_s(z, w_\parallel, w_\perp, t) := \frac{v_{ts}^3(z,t)}{n_s(z,t)} f_s\Big(z, u_{s\parallel}(z,t) + v_{ts}(z,t) w_\parallel, v_{ts}(z,t) w_\perp, t\Big), \tag{3.1}$$

where we have defined the normalized velocities

$$w_\parallel(z, v_\parallel, t) := \frac{v_\parallel - u_{s\parallel}(z,t)}{v_{ts}(z,t)} \tag{3.2}$$

and

$$w_\perp(z, v_\perp, t) := \frac{v_\perp}{v_{ts}(z,t)}, \tag{3.3}$$

the average parallel velocity

$$u_{s\parallel}(z,t) := \frac{2\pi}{n_s} \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \, v_\perp v_\parallel f_s(z, v_\parallel, v_\perp, t) \tag{3.4}$$

and the thermal speed

$$v_{ts}(z,t) := \sqrt{\frac{4\pi}{3n_s} \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \, v_\perp [(v_\parallel - u_{s\parallel}(z,t))^2 + v_\perp^2] f_s(z, v_\parallel, v_\perp, t)}. \tag{3.5}$$

According to its definition, $F_s(z, w_\parallel, w_\perp, t)$ must satisfy the conditions

$$2\pi \int_{-\infty}^{\infty} \mathrm{d}w_\parallel \int_0^\infty \mathrm{d}w_\perp \, w_\perp F_s(z, w_\parallel, w_\perp, t) = 1, \tag{3.6}$$

$$2\pi \int_{-\infty}^{\infty} \mathrm{d}w_\parallel \int_0^\infty \mathrm{d}w_\perp \, w_\perp w_\parallel F_s(z, w_\parallel, w_\perp, t) = 0 \tag{3.7}$$

and

$$2\pi \int_{-\infty}^{\infty} \mathrm{d}w_\parallel \int_0^\infty \mathrm{d}w_\perp \, w_\perp (w_\parallel^2 + w_\perp^2) F_s(z, w_\parallel, w_\perp, t) = \frac{3}{2} \tag{3.8}$$

at every point z and time t.

The equations for ions become

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial z}\left(n_i u_{i\parallel}\right) = 0, \tag{3.9}$$

$$n_i m_i \left(\frac{\partial u_{i\parallel}}{\partial t} + u_{i\parallel} \frac{\partial u_{i\parallel}}{\partial z}\right) = -\frac{\partial p_{i\parallel}}{\partial z} - e n_i \frac{\partial \phi}{\partial z} + n_i n_n m_i R_{in}(u_{n\parallel} - u_{i\parallel}), \tag{3.10}$$

$$\frac{3}{2}n_i m_i v_{ti}\left(\frac{\partial v_{ti}}{\partial t}+u_{i\|}\frac{\partial v_{ti}}{\partial z}\right)=-\frac{\partial q_{i\|}}{\partial z}-p_{i\|}\frac{\partial u_{i\|}}{\partial z}+\frac{3}{4}n_i n_n m_i R_{in}(v_{tn}^2-v_{ti}^2)$$
$$+\frac{1}{2}n_i n_n m_i R_{in}(u_{n\|}-u_{i\|})^2 \qquad (3.11)$$

and

$$\frac{\partial F_i}{\partial t}+\dot{z}_i\frac{\partial F_i}{\partial z}+\dot{w}_{\|i}\frac{\partial F_i}{\partial w_\|}+\dot{w}_{\perp i}\frac{\partial F_i}{\partial w_\perp}=\dot{F}_i+\mathcal{C}_{in}. \qquad (3.12)$$

Here, we have defined the coefficients

$$\dot{z}_s[F_s](z,w_\|,t):=u_{s\|}+v_{ts}w_\|, \qquad (3.13)$$

$$\dot{w}_{\|s}[F_s](z,w_\|,t):=\frac{1}{n_s m_s v_{ts}}\frac{\partial p_{s\|}}{\partial z}+\frac{2w_\|}{3n_s m_s v_{ts}^2}\left[\frac{\partial q_{s\|}}{\partial z}+\left(p_{s\|}-\frac{3}{2}n_s m_s v_{ts}^2\right)\frac{\partial u_{s\|}}{\partial z}\right]$$
$$-w_\|^2\frac{\partial v_{ts}}{\partial z}, \qquad (3.14)$$

$$\dot{w}_{\perp s}[F_s](z,w_\|,w_\perp,t):=\frac{2w_\perp}{3n_s m_s v_{ts}^2}\left(\frac{\partial q_{s\|}}{\partial z}+p_{s\|}\frac{\partial u_{s\|}}{\partial z}\right)-w_\| w_\perp\frac{\partial v_{ts}}{\partial z} \qquad (3.15)$$

and

$$\dot{F}_s[F_s](z,w_\|,w_\perp,t):=\left[w_\|\left(3\frac{\partial v_{ts}}{\partial z}-\frac{v_{ts}}{n_s}\frac{\partial n_s}{\partial z}\right)\right.$$
$$\left.-\frac{2}{n_s m_s v_{ts}^2}\left(\frac{\partial q_{s\|}}{\partial z}+\left(p_{s\|}-\frac{1}{2}n_s m_s v_{ts}^2\right)\frac{\partial u_{s\|}}{\partial z}\right)\right]F_s, \qquad (3.16)$$

the parallel pressure

$$p_{s\|}[F_s](z,t):=2\pi n_s m_s v_{ts}^2\int_{-\infty}^{\infty}\mathrm{d}w_\|\int_0^\infty \mathrm{d}w_\perp\, w_\perp w_\|^2 F_s(z,w_\|,w_\perp,t) \qquad (3.17)$$

the parallel heat flux

$$q_{s\|}[F_s](z,t):=\pi n_s m_s v_{ts}^3\int_{-\infty}^{\infty}\mathrm{d}w_\|\int_0^\infty \mathrm{d}w_\perp\, w_\perp w_\|(w_\|^2+w_\perp^2)F_s(z,w_\|,w_\perp,t), \qquad (3.18)$$

and the modified charge exchange collision operator

$$\mathcal{C}_{in}[F_i,F_n,n_n,u_{i\|},u_{n\|},v_{ti},v_{tn}](z,w_\|,w_\perp,t)$$
$$:=-n_n R_{in}\left[F_i-\frac{v_{ti}^3}{v_{tn}^3}F_n\left(z,\frac{u_{i\|}-u_{n\|}}{v_{tn}}+\frac{v_{ti}}{v_{tn}}w_\|,\frac{v_{ti}}{v_{tn}}w_\perp,t\right)\right]$$
$$+n_n R_{in}\frac{\partial}{\partial w_\|}\left[\left(\frac{u_{n\|}-u_{i\|}}{v_{ti}}+\frac{w_\|}{2}\left(\frac{v_{tn}^2}{v_{ti}^2}-1+\frac{2(u_{n\|}-u_{i\|})^2}{3v_{ti}^2}\right)\right)F_i\right]$$
$$+\frac{n_n R_{in}}{w_\perp}\frac{\partial}{\partial w_\perp}\left[\frac{w_\perp^2}{2}\left(\frac{v_{tn}^2}{v_{ti}^2}-1+\frac{2(u_{n\|}-u_{i\|})^2}{3v_{ti}^2}\right)F_i\right]. \qquad (3.19)$$

Note that the differential terms in this modified collision operator could have been included in the definitions of the coefficients $\dot{w}_{\|i}$, $\dot{w}_{\perp i}$ and $\dot{F}_i$, but we have decided to make them part of a modified collision operator instead to separate the effect of collisions clearly. This split should not be taken as a suggestion on how to implement these terms in a code.

The equations for the neutrals are

$$\frac{\partial n_n}{\partial t} + \frac{\partial}{\partial z}\left(n_n u_{n\parallel}\right) = 0, \tag{3.20}$$

$$n_n m_i \left(\frac{\partial u_{n\parallel}}{\partial t} + u_{n\parallel}\frac{\partial u_{n\parallel}}{\partial z}\right) = -\frac{\partial p_{n\parallel}}{\partial z} + n_i n_n m_i R_{in}(u_{i\parallel} - u_{n\parallel}), \tag{3.21}$$

$$\frac{3}{2} n_n m_i v_{tn}\left(\frac{\partial v_{tn}}{\partial t} + u_{n\parallel}\frac{\partial v_{tn}}{\partial z}\right) = -\frac{\partial q_{n\parallel}}{\partial z} - p_{n\parallel}\frac{\partial u_{n\parallel}}{\partial z} + \frac{3}{4} n_n n_i m_i R_{in}(v_{ti}^2 - v_{tn}^2) + \frac{1}{2} n_n n_i m_i R_{in}(u_{n\parallel} - u_{i\parallel})^2 \tag{3.22}$$

and

$$\frac{\partial F_n}{\partial t} + \dot{z}_n\frac{\partial F_n}{\partial z} + \dot{w}_{\parallel n}\frac{\partial F_n}{\partial w_\parallel} + \dot{w}_{\perp n}\frac{\partial F_n}{\partial w_\perp} = \dot{F}_n + \mathcal{C}_{ni}. \tag{3.23}$$

Here, we have defined the modified charge exchange collision operator

$$\begin{aligned}
&\mathcal{C}_{ni}[F_n, F_i, n_i, u_{n\parallel}, u_{i\parallel}, v_{tn}, v_{ti}](z, w_\parallel, w_\perp, t) \\
&\quad := -n_i R_{in}\left[F_n - \frac{v_{tn}^3}{v_{ti}^3} F_i\left(z, \frac{u_{n\parallel} - u_{i\parallel}}{v_{ti}} + \frac{v_{tn}}{v_{ti}} w_\parallel, \frac{v_{tn}}{v_{ti}} w_\perp, t\right)\right] \\
&\quad\quad + n_i R_{in}\frac{\partial}{\partial w_\parallel}\left[\left(\frac{u_{i\parallel} - u_{n\parallel}}{v_{tn}} + \frac{w_\parallel}{2}\left(\frac{v_{ti}^2}{v_{tn}^2} - 1 + \frac{2(u_{n\parallel} - u_{i\parallel})^2}{3v_{tn}^2}\right)\right) F_n\right] \\
&\quad\quad + \frac{n_i R_{in}}{w_\perp}\frac{\partial}{\partial w_\perp}\left[\frac{w_\perp^2}{2}\left(\frac{v_{ti}^2}{v_{tn}^2} - 1 + \frac{2(u_{n\parallel} - u_{i\parallel})^2}{3v_{tn}^2}\right) F_n\right].
\end{aligned} \tag{3.24}$$

Equations (3.12) and (3.23) for F_i and F_n are constructed such that conditions (3.6), (3.7) and (3.8) are satisfied at all times if they are satisfied at $t = 0$.

4. Linear test

One possible test for the sets of 1D equations described above is the evolution of small perturbations to a uniform Maxwellian equilibrium. We assume the following form for the ion and neutral distribution functions,

$$f_s(z, v_\parallel, v_\perp, t) = f_{Ms}(v_\parallel, v_\perp) + f_{s1}(v_\parallel, v_\perp)[\exp(\mathrm{i}k_\parallel z - \mathrm{i}\omega t) + \text{complex conjugate}], \tag{4.1}$$

where

$$f_{Ms}(v_\parallel, v_\perp) = n_s\left(\frac{m_i}{2\pi T_h}\right)^{3/2}\exp\left(-\frac{m_i(v_\parallel^2 + v_\perp^2)}{2T_h}\right). \tag{4.2}$$

Note that both species share the same constant temperature T_h. To ensure that the potential is small, we assume $n_i = N_e$.

Since the perturbations $f_{s1}(v_\parallel, v_\perp)$ and ϕ are small, equations (2.14), (2.15) and (2.16) can be linearized to give

$$(k_\parallel v_\parallel - \omega - \mathrm{i}n_n R_{in}) f_{i1} + \mathrm{i}n_i R_{in} f_{n1} = -\frac{e\phi}{T_h} k_\parallel v_\parallel\, f_{Mi} + \mathrm{i}R_{in}(n_{n1} f_{Mi} - n_{i1} f_{Mn}), \tag{4.3}$$

$$\mathrm{i}n_n R_{in} f_{i1} + (k_\parallel v_\parallel - \omega - \mathrm{i}n_i R_{in}) f_{n1} = \mathrm{i}R_{in}(n_{i1} f_{Mn} - n_{n1} f_{Mi}) \tag{4.4}$$

and

$$\frac{n_{i1}}{n_i} = \frac{e\phi}{T_e}. \tag{4.5}$$

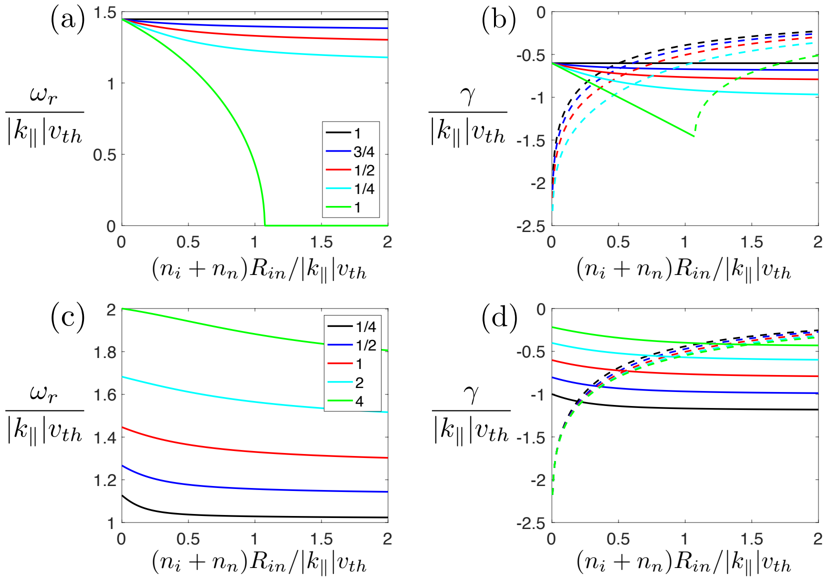

FIGURE 1. Solutions to the dispersion relation (4.7): acoustic waves (solid lines) and non-propagating modes (dashed lines). (a) Real frequency $\omega_r := \mathrm{Re}(\omega)$ and (b) growth rate $\gamma := \mathrm{Im}(\omega)$ as functions of the charge exchange collision frequency $(n_i + n_n)R_{in}$ for $T_e/T_h = 1$ and several values of the parameter $n_i/(n_i + n_n)$. (c) Real frequency ω_r and (d) growth rate γ as functions of the the charge exchange collision frequency $(n_i + n_n)R_{in}$ for $n_i/(n_i + n_n) = 1/2$ and several values of the parameter T_e/T_h.

Here, we have defined the perturbations to the density as

$$n_{s1} = 2\pi \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \, v_\perp f_{s1}. \tag{4.6}$$

Solving for the functions f_{i1} and f_{n1} as functions of n_{i1} and n_{n1} and then integrating f_{i1} and f_{n1} over velocity space, we find the equations

$$\begin{pmatrix} A_{ii} & A_{in} \\ A_{ni} & A_{nn} \end{pmatrix} \begin{pmatrix} n_{i1} \\ n_{n1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.7}$$

where the elements of the matrix are

$$A_{ii} = 1 + \frac{T_e}{T_h} + \frac{n_i}{n_i + n_n}\frac{T_e}{T_h}\zeta Z(\zeta) + \frac{n_n}{n_i + n_n}\left[\left(1 + \frac{T_e}{T_h}\right)\zeta_{in} - \zeta\right] Z(\zeta_{in}) \tag{4.8}$$

$$A_{in} = -\frac{n_i}{n_i + n_n}(\zeta_{in} - \zeta) Z(\zeta_{in}) \tag{4.9}$$

$$A_{ni} = -\frac{n_n}{n_i + n_n}\left\{\left[\left(1 + \frac{T_e}{T_h}\right)\zeta_{in} - \zeta\right] Z(\zeta_{in}) - \frac{T_e}{T_h}\zeta Z(\zeta)\right\} \tag{4.10}$$

and

$$A_{nn} = 1 + \frac{n_i}{n_i + n_n}(\zeta_{in} - \zeta) Z(\zeta_{in}). \tag{4.11}$$

Here, we have defined

$$\zeta := \frac{\omega}{|k_\parallel| v_{th}}, \quad \zeta_{in} := \frac{\omega + \mathrm{i}(n_i + n_n)R_{in}}{|k_\parallel| v_{th}}, \tag{4.12}$$

with $v_{th} := \sqrt{2T_h/m_i}$, and we have used the plasma dispersion function (Fried & Conte

1961)

$$Z(\zeta) := \exp(-\zeta^2)\left(\mathrm{i}\sqrt{\pi} - 2\int_0^{\zeta} \exp(y^2)\,\mathrm{d}y\right). \tag{4.13}$$

By setting the determinant of the matrix in equation (4.7) to zero, we can calculate the frequency ω of the modes for an initial $k_{\|}$.

We show two different types of solutions to the dispersion relation in figure 1: acoustic waves that have both real frequency $\omega_r := \mathrm{Re}(\omega)$ and damping rate $\gamma := \mathrm{Im}(\omega)$, and non-propagating modes with $\omega_r = 0$. In the figure, we plot the real frequency and damping rate for the acoustic waves as solid lines, whereas for the non-propagating modes, we only plot the damping rates as dashed lines. We can use these solutions to benchmark the implementation of the equations in our code.

5. Conclusions

We have identified the first set of equations that we will use to test a new approach to drift kinetics that extracts the low order moments from the distribution function. The chosen model can be integrated without employing implicit time-stepping methods. This is a choice that we have made to ensure that we can compare the new model with the well-established drift kinetic model.

We have also developed an analytical benchmark for the equations. The calculation ignores ion-ion collisions and it is hence not relevant to all edge operational regimes, but it allows us to test the implementation of the equations with and without collisions. Similar calculations can be performed including the full ion-ion collision operator and model collision operators.

REFERENCES

Braginskii, S.I. 1958 Transport phenomena in a completely ionized two-temperature plasma. *Sov. Phys. JETP.* **6**, 358.

Catto, P.J. 1994 A short mean-free path, coupled neutral-ion transport description of a tokamak edge plasma. *Phys. Plasmas* **1**, 1936.

Connor, J.W. 1977 An analytic solution for the distribution function of neutral particles in a Maxwellian plasma using the method of singular eignfunctions. *Plasma Phys.* **19**, 853.

Fried, B.D. & Conte, S.D. 1961 *The Plasma Dispersion Function: The Hilbert transform of the Gaussian*. Academic Press.

Hazeltine, R.D. 1973 Recursive derivation of drift-kinetic equation. *Plasma Phys.* **15**, 77–80.

Hazeltine, R.D., Calvin, M.D., Valanju, P.M. & Solano, E.R. 1992 Analytical calculation of neutral transport and its effect on ions. *Nucl. Fusion* **32**, 3.

Rosenbluth, M.N., MacDonald, W.M. & Judd, D.L. 1957 Fokker-Planck Equation for an Inverse-Square Force. *Phys. Rev.* **107**, 1.

Appendix A. The Maxwell-Boltzmann response

The Maxwell-Boltzmann response in equation (2.13) is the solution to electron drift kinetic equation (2.4) in the limit $\sqrt{m_e/m_i} \ll 1$. The expansion in the mass ratio is based on the fact that the species within the plasma tend to thermalize due to collisions, and hence the different species have in general similar average kinetic energies. Thus, the characteristic thermal speeds of the ions and neutrals, v_{ti} and v_{tn}, scale as $m_i^{-1/2}$, whereas the electron thermal speed scales as $m_e^{-1/2}$, giving $v_{te} \gg v_{ti} \sim v_{tn}$.

We assume that the massive ions and neutrals control the dynamics of interest, giving the estimate

$$\frac{\partial}{\partial t} \sim \frac{v_{ti}}{L}. \tag{A 1}$$

Thus, the time derivative in equation (2.4) is negligible compared to terms like

$$v_\parallel \frac{\partial f_e}{\partial z} \sim f_e \frac{v_{te}}{L}. \tag{A 2}$$

Hence, we can neglect the time derivative to find

$$v_\parallel \frac{\partial f_e}{\partial z} + \frac{e}{m_e}\frac{\partial \phi}{\partial z}\frac{\partial f_e}{\partial v_\parallel} = C_{ee}[f_e] + C_{ei}[f_e, f_i] + C_{en}[f_e, f_n]. \tag{A 3}$$

To solve this equation, we need to use the properties of the collision operators. The electron-electron collision operator satisfies an H-theorem: the entropy production

$$-\int \ln f_e \, C_{ee}[f_e] \, \mathrm{d}^3 v \geqslant 0 \tag{A 4}$$

is always positive and it only vanishes if f_e is a Maxwellian. The elastic collision operators $C_{ei}[f_e, f_i]$ and $C_{en}[f_e, f_n]$ also satisfy H-theorems, but they are much more complicated as in general these theorems involve the ions and the neutrals. Luckily, if we perform the expansion $\sqrt{m_e/m_i} \ll 1$, $C_{ei}[f_e, f_i]$ and $C_{en}[f_e, f_n]$ satisfy simplified versions of their H-theorems, namely, the entropy productions

$$-\int \ln f_e \, C_{ei}[f_e, f_i] \, \mathrm{d}^3 v \geqslant 0 \text{ and } -\int \ln f_e \, C_{en}[f_e, f_n] \, \mathrm{d}^3 v \geqslant 0 \tag{A 5}$$

are always positive, and they only vanish if f_e is isotropic. Note that, in the limit $\sqrt{m_e/m_i} \ll 1$, these operators do not impose conditions on f_i or f_n.

Armed with these properties, we multiply equation (A 3) by $-\ln f_e$ and we integrate over velocity space to obtain

$$\frac{\partial}{\partial z}\left[-\int (f_e \ln f_e - f_e) \, v_\parallel \, \mathrm{d}^3 v\right] = -\int \ln f_e \, C_{ee}[f_e] \, \mathrm{d}^3 v - \int \ln f_e \, C_{ei}[f_e, f_i] \, \mathrm{d}^3 v \\ - \int \ln f_e \, C_{en}[f_e, f_n] \, \mathrm{d}^3 v. \tag{A 6}$$

Integrating this equation over z and using the periodic boundary conditions, we finally obtain

$$0 = -\int_0^L \mathrm{d}z \int \ln f_e \, C_{ee}[f_e] \, \mathrm{d}^3 v - \int_0^L \mathrm{d}z \int \ln f_e \, C_{ei}[f_e, f_i] \, \mathrm{d}^3 v \\ - \int_0^L \mathrm{d}z \int \ln f_e \, C_{en}[f_e, f_n] \, \mathrm{d}^3 v. \tag{A 7}$$

Since the entropy production of each collision operator is always positive, this equation can only be satisfied if each of the entropy productions vanish at every z. This implies that, at every z, f_e is Maxwellian and isotropic,

$$f_e(z, v_\parallel, v_\perp) = f_{Me}(z, v_\parallel, v_\perp, t) := n_e(z,t)\left(\frac{m_e}{2\pi T_e(z,t)}\right)^{3/2} \exp\left(-\frac{m_e(v_\parallel^2 + v_\perp^2)}{2T_e(z,t)}\right). \tag{A 8}$$

We need to determine the dependence of $n_e(z,t)$ and $T_e(z,t)$ on z. Substituting the solution $f_{Me}(z, v_\parallel, v_\perp, t)$ on equation (A 3), we find

$$\left[v_\parallel \left(\frac{\partial}{\partial z} \ln n_e - \frac{e}{T_e} \frac{\partial \phi}{\partial z} \right) + v_\parallel \left(\frac{m_e(v_\parallel^2 + v_\perp^2)}{2T_e} - \frac{3}{2} \right) \frac{\partial}{\partial z} \ln T_e \right] f_{Me} = 0. \tag{A 9}$$

Since this equation has to be satisfied for every value of $v_\parallel$ and $v_\perp$, $T_e(t)$ cannot depend on z, and $n_e(z,t) = N_e(t) \exp(e\phi(z,t)/T_e(t))$. Thus, we find equation (2.13) with N_e and T_e being in general functions of t. For simplicity, we choose them to be constants.