1D drift kinetic models with wall boundary conditions

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1. Introduction

In previous reports, we proposed 1D drift kinetic equations with periodic boundary conditions, adequate for the closed field line region of the edge. In this report, we discuss a minimal 1D drift kinetic model with wall boundary conditions that represents open field lines. The basic drift kinetic model is presented in section 2, and the wall boundary conditions are discussed in section 3. We then proceed to determine the novel moment drift kinetic equations for ions and neutrals in section 4, and for electrons in section 5. The electrons have to be treated differently so that we can exploit the expansion in the electron-ion mass ratio. We finish with a discussion in section 6. Some of the details of the collision operators in the moment drift kinetic formulation are relegated to appendices to make the main text easier to read.

2. 1D electrostatic drift kinetics

We consider a plasma with one ion species with charge e and mass m_i, electrons with charge $-e$ and mass m_e, and one species of neutrals with mass

$$m_n = m_i. \tag{2.1}$$

The plasma is magnetized by a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, and we assume that the plasma only varies along magnetic field lines. In this case, the electric field produced by the plasma is electrostatic, $\mathbf{E} = -(\partial\phi/\partial z)\hat{\mathbf{z}}$. The potential $\phi(z,t)$ depends on the position along magnetic field lines z and on time t.

If we assume that the gyroradii are small compared to the length scales of interest, and that the gyrofrequencies are much larger than the frequencies that we want to model (Hazeltine 1973), the distribution functions $f_s(z, v_\parallel, v_\perp, t)$ of the different species $s = i, e, n$ only depend on the component of the velocity parallel to the magnetic field $v_\parallel$ and the magnitude of the velocity perpendicular to the magnetic field $v_\perp$, and are independent of the direction of the velocity perpendicular to the magnetic field. Thus, the distribution functions that in general can depend on three spatial variables $\mathbf{r}$, three components of the velocity $\mathbf{v}$ and the time t depend only on z, $v_\parallel$, $v_\perp$ and t,

$$f_s(\mathbf{r}, \mathbf{v}, t) = f_s(z, v_\parallel, v_\perp, t). \tag{2.2}$$

The equations for the distribution functions of the different species are

$$\frac{\partial f_i}{\partial t} + v_\parallel \frac{\partial f_i}{\partial z} - \frac{e}{m_i} \frac{\partial \phi}{\partial z} \frac{\partial f_i}{\partial v_\parallel} = C_{ii}[f_i] + C_{in}[f_i, f_n] + C_{i,\mathrm{ion}}[f_e, f_n] + C_{ie}[f_i, f_e] + S_i, \tag{2.3}$$

$$\frac{\partial f_e}{\partial t} + v_\parallel \frac{\partial f_e}{\partial z} + \frac{e}{m_e}\frac{\partial \phi}{\partial z}\frac{\partial f_e}{\partial v_\parallel} = C_{ee}[f_e] + C_{ei}[f_e, f_i]\left[1 + O\left(\frac{m_e}{m_i}\right)\right]$$

$$+ C_{en}[f_e, f_n]\left[1 + O\left(\frac{m_e}{m_i}\right)\right] + C_{e,\text{ion}}[f_e, f_n] + S_e \qquad (2.4)$$

and

$$\frac{\partial f_n}{\partial t} + v_\parallel \frac{\partial f_n}{\partial z} = C_{ni}[f_n, f_i] + C_{ne}[f_n, f_e] + C_{n,\text{ion}}[f_n, f_e] + S_n. \qquad (2.5)$$

The sources $S_s(z, w_\parallel, w_\perp, t)$ with $s = i, e, n$ represent heating, fueling and the effect of transport perpendicular to the magnetic field line.

We have included the following collisions.

• Ion-ion and electron-electron collisions are modeled by the Fokker-Planck collision operators $C_{ii}[f_i]$ and $C_{ee}[f_e]$ (Rosenbluth *et al.* 1957),

$$C_{ss}[f_s] := \frac{2\pi e^4 \ln \Lambda}{(4\pi\epsilon_0)^2 m_s^2}\nabla_v \cdot (\mathbf{D}[f_s]\cdot\nabla_v f_s + \mathbf{P}[f_s]f_s), \qquad (2.6)$$

where the matrix $\mathbf{D}$ is

$$\mathbf{D}[f_s] := \int \frac{|\mathbf{v}-\mathbf{v}'|^2\mathbf{I} - (\mathbf{v}-\mathbf{v}')(\mathbf{v}-\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|^3}f_s(\mathbf{v}')\,\mathrm{d}^3v' \qquad (2.7)$$

and the vector $\mathbf{P}$ is

$$\mathbf{P}[f_s] := -2\int \frac{\mathbf{v}-\mathbf{v}'}{|\mathbf{v}-\mathbf{v}'|^3}f_s(\mathbf{v}')\,\mathrm{d}^3v'. \qquad (2.8)$$

Here, $\mathbf{I}$ is the 3D unit matrix, ϵ_0 the vacuum permittivity and $\ln \Lambda \approx 15$ the Coulomb logarithm.

• The effect of electron-ion and elastic electron-neutral collisions on the electron distribution function can be simplified in the limit of small electron-ion mass ratio, $m_e/m_i \ll 1$. With this expansion, we find the simplified Fokker-Planck collision operator

$$C_{ei}[f_e, f_i] := \frac{2\pi e^4 n_i \ln \Lambda}{(4\pi\epsilon_0)^2 m_e^2}\nabla_v \cdot \left[\frac{|\mathbf{v}-\mathbf{u}_i|^2\mathbf{I} - (\mathbf{v}-\mathbf{u}_i)(\mathbf{v}-\mathbf{u}_i)}{|\mathbf{v}-\mathbf{u}_i|^3}\cdot\nabla_v f_e\right] \qquad (2.9)$$

for electron-ion collisions (Braginskii 1958), and the simplified Boltzmann collision operator

$$C_{en}[f_e, f_n] := \frac{n_n}{4\pi}\int_0^\pi \mathrm{d}\chi \int_0^{2\pi} \mathrm{d}\varphi \, \sin\chi R_{en}(|\mathbf{v}-\mathbf{u}_n|, \chi)\left[f_e(\overline{\mathbf{v}}(\mathbf{v}, \chi, \varphi, \mathbf{u}_n)) - f_e(\mathbf{v})\right] \qquad (2.10)$$

for electron-neutral collisions. Here

$$n_s(z, t) := 2\pi \int_{-\infty}^\infty \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \, v_\perp f_s(z, v_\parallel, v_\perp, t). \qquad (2.11)$$

is the density of species s, $\mathbf{u}_s := n_s^{-1}\int \mathbf{v}f_s\,\mathrm{d}^3v$ is the average velocity of species s,

$$\overline{\mathbf{v}}(\mathbf{v}, \chi, \varphi, \mathbf{u}_n) := \mathbf{u}_n + \cos\chi(\mathbf{v}-\mathbf{u}_n) + |\mathbf{v}-\mathbf{u}_n|\sin\chi(\cos\varphi\,\hat{\mathbf{e}}_1 + \sin\varphi\,\hat{\mathbf{e}}_2) \qquad (2.12)$$

is a rotation of the vector $\mathbf{v}$ centered around $\mathbf{u}_n$, $R_{en}(|\mathbf{v}-\mathbf{u}_n|, \chi)$ is a function determined by the physics of the electron-neutral collisions, and the unit vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are chosen to form an orthonormal basis with the vector $(\mathbf{v}-\mathbf{u}_n)/|\mathbf{v}-\mathbf{u}_n|$. In equation (2.4), we have indicated that both C_{ei} and C_{en} are missing pieces small in m_e/m_i. These pieces can become important because they represent collisional energy exchange and collisional heating, but they are cumbersome. The moment method that we propose in

this document will allow us to keep these important effects even with the simplified collision operators (2.9) and (2.10).

• The expansion in electron-ion mass ratio also implies electron-ion collisions and electron-neutral collisions have a very small effect on f_i and f_n – the terms C_{ie} and C_{ne} in equations (2.3) and (2.5) are small compared with C_{ii} and C_{ni} by a factor of $\sqrt{m_e/m_i} \ll 1$,

$$C_{ie} \sim \sqrt{\frac{m_e}{m_i}} C_{ii}, \quad C_{ne} \sim \sqrt{\frac{m_e}{m_i}} C_{ni}. \tag{2.13}$$

Like the mass ratio corrections to C_{ei} and C_{en}, these terms can become important because they contain the collisional energy exchange between electrons and the heavier species. We will keep these effects in a simplified form in our moment formulation.

• Charge-exchange collisions are represented by the simplified Boltzmann collision operators

$$C_{in}[f_i, f_n] := -\int R_{in}(|\mathbf{v} - \mathbf{v}'|) [f_i(\mathbf{v})f_n(\mathbf{v}') - f_i(\mathbf{v}')f_n(\mathbf{v})] \, \mathrm{d}^3 v' \tag{2.14}$$

and

$$C_{ni}[f_n, f_i] := -\int R_{in}(|\mathbf{v} - \mathbf{v}'|) [f_n(\mathbf{v})f_i(\mathbf{v}') - f_n(\mathbf{v}')f_i(\mathbf{v})] \, \mathrm{d}^3 v'. \tag{2.15}$$

• To model ionization, we use the collision operators

$$C_{i,\mathrm{ion}}[f_e, f_n] := f_n \int R_{\mathrm{ion}}(v') f_e(\mathbf{v}') \, \mathrm{d}^3 v' \tag{2.16}$$

and

$$C_{n,\mathrm{ion}}[f_e, f_n] := -f_n \int R_{\mathrm{ion}}(v') f_e(\mathbf{v}') \, \mathrm{d}^3 v'. \tag{2.17}$$

We also need to include a collision operator $C_{e,\mathrm{ion}}$ in the electron equation to model the increase in the number of electrons and the energy loss due to ionization. This operator is complicated because it involves three particles (the resulting ion and two electrons), but we will be able to avoid giving it a definite form. Instead, we will use the expansion in $m_e/m_i \ll 1$ and the fact that

$$C_{e,\mathrm{ion}}[f_e, f_n] \sim n_n R_{\mathrm{ion}} f_e. \tag{2.18}$$

• We have neglected neutral-neutral collisions because, in current fusion devices, the neutral density is sufficiently small that the neutral-neutral collisions are rare. It is possible that the higher densities expected in fusion reactors will make neutral-neutral collisions more relevant. To include neutral-neutral collisions, a Boltzmann collision operator is in principle required, but using a simplified collision may be possible if the exact shape of the neutral distribution function is not important for the physics of interest.

To simplify our equations, we assume that the functions R_{en}, R_{in} and R_{ion} are constant (Connor 1977; Hazeltine *et al.* 1992; Catto 1994), finding

$$C_{en}[f_e, f_n] = n_n R_{en} \left[\frac{1}{2} \int_0^\pi f_e(z, u_{n\parallel} + |\mathbf{v} - \mathbf{u}_n| \cos \chi, |\mathbf{v} - \mathbf{u}_n| \sin \chi, t) \sin \chi \, \mathrm{d}\chi \right.$$
$$\left. - f_e(z, v_\parallel, v_\perp, t) \right], \tag{2.19}$$

with $|\mathbf{v} - \mathbf{u}_n| = \sqrt{(v_\parallel - u_{n\parallel})^2 + v_\perp^2}$,

$$C_{in}[f_i, f_n] = -R_{in}\left(n_n f_i - n_i f_n\right), \tag{2.20}$$

$$C_{ni}[f_n, f_i] = -R_{in}\left(n_i f_n - n_n f_i\right), \tag{2.21}$$

$$C_{i,\text{ion}}[f_e, f_n] = f_n n_e R_{\text{ion}} \tag{2.22}$$

and

$$C_{n,\text{ion}}[f_e, f_n] = -f_n n_e R_{\text{ion}}. \tag{2.23}$$

The potential $\phi(z, t)$ is determined by the quasineutrality equation

$$n_i = n_e. \tag{2.24}$$

To solve this equation, we need to treat the equations implicitly as the potential enters only via its effect on $\partial f_i/\partial t$ and $\partial f_e/\partial t$. The need to use implicit methods is one of the reasons why we are trying to extract some of the low order moments from the distribution function.

3. Wall boundary conditions

The kinetic equations will be solved in the interval $z \in [0, L]$, and we will impose wall boundary conditions at $z = 0$ and $z = L$. We assume the wall to be exactly perpendicular to the magnetic field lines to be able to impose a set of simplified boundary conditions: the logical sheath boundary conditions of Parker *et al.* (1993). When the magnetic field is at an angle to the wall, one needs to consider a thin boundary layer with a width of the order of the ion gyroradius that forms on the wall and is known as the magnetic presheath (Chodura 1982). The complicated boundary conditions that this layer imposes on drift kinetic models are an active area of research (Geraldini *et al.* 2017, 2018, 2019; Geraldini 2021). These works indicate that the magnetic presheath and the Debye sheath must be solved in conjunction with the quasineutral plasma, but this is not necessarily computationally expensive as the presheath and sheath models are 1D or at most 2D, and hence cheaper than the 5D drift kinetic models that one needs for edge turbulence. The technique proposed by Geraldini *et al.* (2018), for example, solves the magnetic presheath in a single processor in seconds, and this technique can be parallelized.

Logical sheath boundary conditions make use of the fact that a thin sheath of non-neutral plasma with a width of the order of the Debye length forms on walls to ensure quasineutrality. The potential drop across this sheath repels electrons away from the wall because otherwise electrons would flow to the wall at much greater rate than ions due to their lower mass and higher thermal speed. In our model, $\phi(0, t)$ and $\phi(L, t)$ are not the potential of the wall, but the potential at the entrance of the sheath. In this report, we choose the potential of the wall at $z = 0$ to be 0 without loss of generality. We denote the potential of the wall at $z = L$ as ϕ_w. Then, for the sheaths to repel electrons, $\phi(0, t)$ must be larger than 0 and $\phi(L, t)$ must be larger than ϕ_w.

The value of the potential at $z = 0$ and $z = L$ is determined by requiring that the current towards the wall at both $z = 0$ and $z = L$ vanishes. We consider the sheath at $z = L$ first, and we will then apply the results that we obtain to the sheath at $z = 0$. Since the thin sheath at $z = L$ imposes a large electric field perpendicular to the wall, which in this case is along the magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, the sheath only modifies the parallel velocity of electrons. Within the sheath, the parallel energy $\mathcal{E}_\parallel := m_e v_\parallel^2/2 - e\phi$ is conserved, and as a result an electron that has velocity $v_\parallel$ at the entrance of the sheath

is slowed down to a parallel velocity $\sqrt{v_\parallel^2 - 2e(\phi(L,t) - \phi_w)/m_e}$ when it reaches the wall. Thus, electrons with parallel velocity larger than $\sqrt{2e(\phi(L,t) - \phi_w)/m_e}$ reach the wall, where they recombine with ions, whereas electrons with parallel velocity smaller than $\sqrt{2e(\phi(L,t) - \phi_w)/m_e}$ are repelled back into the quasineutral plasma. Thus, the boundary condition on the electron distribution function at $z = L$ is

$$f_e(L, v_\parallel < 0, v_\perp, t) = \begin{cases} f_e(L, -v_\parallel, v_\perp, t) & \text{for } v_\parallel \geqslant -\sqrt{2e(\phi(L,t) - \phi_w)/m_e}, \\ 0 & \text{for } v_\parallel < -\sqrt{2e(\phi(L,t) - \phi_w)/m_e}, \end{cases} \quad (3.1)$$

i.e. the electron distribution is mirrored with respect to $v_\parallel = 0$ and a cut-off is imposed for sufficiently negative parallel velocities. Note that no boundary condition is needed for $v_\parallel > 0$ at $z = L$ because of the direction of the characteristics of the kinetic equation. Expression (3.1) also gives the electron current density towards the wall at the entrance of the sheath at $z = L$,

$$J_{e\parallel}(L,t) = -2\pi e \int_{\sqrt{2e(\phi(L,t) - \phi_w)/m_e}}^{\infty} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \, v_\perp v_\parallel f_e(L, v_\parallel, v_\perp, t). \quad (3.2)$$

This is the electron current density at the wall because electron flow is conserved across the sheath. Imposing that the electron current cancels the ion current gives us a nonlinear equation for the potential difference $\phi(L,t) - \phi_w$,

$$2\pi \int_{\sqrt{2e(\phi(L,t) - \phi_w)/m_e}}^{\infty} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \, v_\perp v_\parallel f_e(L, v_\parallel, v_\perp, t)$$

$$= 2\pi \int_0^\infty \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \, v_\perp v_\parallel f_i(L, v_\parallel, v_\perp, t). \quad (3.3)$$

To obtain the ion current, we have used the fact that the sheath attracts ions and hence no ions can have negative parallel velocity at the entrance of the sheath at $z = L$.

The conditions at $z = 0$ for the electron distribution and the potential are similar to those for $z = L$. For the electron distribution function, we find

$$f_e(0, v_\parallel > 0, v_\perp, t) = \begin{cases} f_e(0, -v_\parallel, v_\perp, t) & \text{for } v_\parallel \leqslant \sqrt{2e\phi(0,t)/m_e}, \\ 0 & \text{for } v_\parallel > \sqrt{2e\phi(0,t)/m_e}, \end{cases} \quad (3.4)$$

and for the potential we obtain

$$2\pi \int_{-\infty}^{-\sqrt{2e\phi(0,t)/m_e}} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \, v_\perp v_\parallel f_e(0, v_\parallel, v_\perp, t)$$

$$= 2\pi \int_{-\infty}^0 \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \, v_\perp v_\parallel f_i(0, v_\parallel, v_\perp, t), \quad (3.5)$$

Note that conditions (3.3) and (3.5) imply that no net electrical current is leaving the system. Thus, the total source of charge in the magnetic field line of interest must be zero,

$$\int_0^L \mathrm{d}z \int S_i \, \mathrm{d}^3 v = \int_0^L \mathrm{d}z \int S_e \, \mathrm{d}^3 v. \quad (3.6)$$

We still need boundary conditions for the ion and neutral distribution functions. Ions recombine when they hit the wall, so no ions come back, giving

$$f_i(0, v_\parallel > 0, v_\perp, t) = 0, \quad f_i(L, v_\parallel < 0, v_\perp, t) = 0. \quad (3.7)$$

The neutrals hit the wall and thermalize at the temperature of the wall T_w, while also

receiving back the ions that have recombined at the wall,

$$f_n(0, v_\| > 0, v_\perp, t) = \Gamma_0 f_{Kw}(v_\|, v_\perp), \quad f_n(L, v_\| < 0, v_\perp, t) = \Gamma_L f_{Kw}(v_\|, v_\perp), \quad (3.8)$$

where

$$\Gamma_0 := \sum_{s=i,n} 2\pi \int_{-\infty}^{0} \mathrm{d}v_\| \int_{0}^{\infty} \mathrm{d}v_\perp \, v_\perp |v_\||f_s(0, v_\|, v_\perp, t) \quad (3.9)$$

and

$$\Gamma_L := \sum_{s=i,n} 2\pi \int_{0}^{\infty} \mathrm{d}v_\| \int_{0}^{\infty} \mathrm{d}v_\perp \, v_\perp v_\| f_s(L, v_\|, v_\perp, t) \quad (3.10)$$

are the fluxes of neutrals and ions towards the walls at $z = 0$ and $z = L$. Here,

$$f_{Kw}(v_\|, v_\perp) := \frac{3}{4\pi} \left(\frac{m_i}{T_w}\right)^2 \frac{|v_\||}{\sqrt{v_\|^2 + v_\perp^2}} \exp\left(-\frac{m_i(v_\|^2 + v_\perp^2)}{2T_w}\right) \quad (3.11)$$

is the Knudsen cosine distribution (Knudsen 1916) that assumes that the particles have entered the wall lattice, have reached thermodynamic equilibrium with it, and have then left the wall. Knudsen showed that this distribution function fits experimental measurements well.

4. 1D moment drift kinetics for ions and neutrals

Instead of solving for $f_s(z, v_\|, v_\perp, t)$ with $s = i, n$, we solve for

$$F_s(z, w_\|, w_\perp, t) := \frac{v_{ts}^3(z, t)}{n_s(z, t)} f_s\left(z, u_{s\|}(z, t) + v_{ts}(z, t)w_\|, v_{ts}(z, t)w_\perp, t\right), \quad (4.1)$$

where we have defined the normalized velocities

$$w_\|(z, v_\|, t) := \frac{v_\| - u_{s\|}(z, t)}{v_{ts}(z, t)} \quad (4.2)$$

and

$$w_\perp(z, v_\perp, t) := \frac{v_\perp}{v_{ts}(z, t)}, \quad (4.3)$$

the average parallel velocity

$$u_{s\|}(z, t) := \frac{2\pi}{n_s} \int_{-\infty}^{\infty} \mathrm{d}v_\| \int_{0}^{\infty} \mathrm{d}v_\perp \, v_\perp v_\| f_s(z, v_\|, v_\perp, t) \quad (4.4)$$

and the thermal speed

$$v_{ts}(z, t) := \sqrt{\frac{2T_s(z, t)}{m_s}}, \quad (4.5)$$

with

$$T_s(z, t) := \frac{2\pi}{n_s} \int_{-\infty}^{\infty} \mathrm{d}v_\| \int_{0}^{\infty} \mathrm{d}v_\perp \, v_\perp \frac{m_s[(v_\| - u_{s\|}(z, t))^2 + v_\perp^2]}{3} f_s(z, v_\|, v_\perp, t) \quad (4.6)$$

the temperature of species s. According to its definition, $F_s(z, w_\|, w_\perp, t)$ must satisfy the conditions

$$2\pi \int_{-\infty}^{\infty} \mathrm{d}w_\| \int_{0}^{\infty} \mathrm{d}w_\perp \, w_\perp F_s(z, w_\|, w_\perp, t) = 1, \quad (4.7)$$

$$2\pi \int_{-\infty}^{\infty} \mathrm{d}w_{\parallel} \int_{0}^{\infty} \mathrm{d}w_{\perp}\, w_{\perp} w_{\parallel} F_s(z, w_{\parallel}, w_{\perp}, t) = 0 \tag{4.8}$$

and

$$2\pi \int_{-\infty}^{\infty} \mathrm{d}w_{\parallel} \int_{0}^{\infty} \mathrm{d}w_{\perp}\, w_{\perp} (w_{\parallel}^2 + w_{\perp}^2) F_s(z, w_{\parallel}, w_{\perp}, t) = \frac{3}{2} \tag{4.9}$$

at every point z and time t.

4.1. *Ion equations*

The equations for n_i, $u_{i\parallel}$ and T_i are

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial z}\left(n_i u_{i\parallel}\right) = n_n n_e R_{\mathrm{ion}} + \int S_i \, \mathrm{d}^3 v, \tag{4.10}$$

$$n_i m_i \left(\frac{\partial u_{i\parallel}}{\partial t} + u_{i\parallel} \frac{\partial u_{i\parallel}}{\partial z}\right) = -\frac{\partial p_{i\parallel}}{\partial z} - e n_i \frac{\partial \phi}{\partial z} + n_i m_i (n_n R_{in} + n_e R_{\mathrm{ion}})(u_{n\parallel} - u_{i\parallel})$$
$$+ \int m_i (v_{\parallel} - u_{i\parallel}) S_i \, \mathrm{d}^3 v \tag{4.11}$$

and

$$\frac{3}{2} n_i \left(\frac{\partial T_i}{\partial t} + u_{i\parallel} \frac{\partial T_i}{\partial z}\right) = -\frac{\partial q_{i\parallel}}{\partial z} - p_{i\parallel} \frac{\partial u_{i\parallel}}{\partial z} + \frac{3}{2} n_i (n_n R_{in} + n_e R_{\mathrm{ion}})(T_n - T_i)$$
$$+ \frac{1}{2} n_i m_i (n_n R_{in} + n_e R_{\mathrm{ion}})(u_{n\parallel} - u_{i\parallel})^2 + \int \frac{1}{2} m_i |\mathbf{v} - u_{i\parallel}\hat{\mathbf{z}}|^2 C_{ie} \, \mathrm{d}^3 v$$
$$+ \int \left(\frac{1}{2} m_i |\mathbf{v} - u_{i\parallel}\hat{\mathbf{z}}|^2 - \frac{3}{2} T_i\right) S_i \, \mathrm{d}^3 v. \tag{4.12}$$

Here, we have defined the parallel pressure

$$p_{s\parallel}[F_s, n_s, v_{ts}](z, t) := 2\pi n_s m_s v_{ts}^2 \int_{-\infty}^{\infty} \mathrm{d}w_{\parallel} \int_{0}^{\infty} \mathrm{d}w_{\perp}\, w_{\perp} w_{\parallel}^2 F_s(z, w_{\parallel}, w_{\perp}, t) \tag{4.13}$$

and the parallel heat flux

$$q_{s\parallel}[F_s, n_s, v_{ts}](z, t) := \pi n_s m_s v_{ts}^3 \int_{-\infty}^{\infty} \mathrm{d}w_{\parallel} \int_{0}^{\infty} \mathrm{d}w_{\perp}\, w_{\perp} w_{\parallel} (w_{\parallel}^2 + w_{\perp}^2) F_s(z, w_{\parallel}, w_{\perp}, t). \tag{4.14}$$

We have also included the term $(1/2)\int m_i |\mathbf{v} - u_{i\parallel}\hat{\mathbf{z}}|^2 C_{ie} \, \mathrm{d}^3 v$ due to collisions with electrons. Collisions with electrons are negligible to lowest order in $\sqrt{m_e/m_i}$ in the ion kinetic equation and thus cannot determine the lowest order distribution function F_i, but when collisions are sufficiently frequent that $\nu_{ii} L/v_{ti} \gtrsim \sqrt{m_i/m_e} \gg 1$, the term $(1/2)\int m_i |\mathbf{v} - u_{i\parallel}\hat{\mathbf{z}}|^2 C_{ie} \, \mathrm{d}^3 v$ becomes comparable to the other terms in the energy equation. Here,

$$\nu_{ii} := \frac{8\sqrt{2\pi}}{3} \frac{e^4 n_i \ln \Lambda}{(4\pi\epsilon_0)^2 m_i^2 v_{ti}^3} \tag{4.15}$$

is the ion-ion collision frequency as defined by Braginskii (Braginskii 1958). At the large collision frequencies required for the term $(1/2)\int m_i |\mathbf{v} - u_{i\parallel}\hat{\mathbf{z}}|^2 C_{ie} \, \mathrm{d}^3 v$ to be relevant, the ion and electron distribution functions become close to a Maxwellian,

$$f_s \simeq f_{Ms} := \frac{n_s}{\pi^{3/2} v_{ts}^3} \exp\left(-\frac{(v_{\parallel} - u_{s\parallel})^2 + v_{\perp}^2}{v_{ts}^2}\right). \tag{4.16}$$

Thus, we can use the approximation

$$\int \frac{1}{2}m_i|\mathbf{v} - u_{i\|}\hat{\mathbf{z}}|^2 C_{ie}[f_i, f_e]\,\mathrm{d}^3 v \simeq \int \frac{1}{2}m_i|\mathbf{v} - u_{i\|}\hat{\mathbf{z}}|^2 C_{ie}[f_{Mi}, f_{Me}]\,\mathrm{d}^3 v$$

$$\simeq \frac{3n_e m_e \nu_{ei}}{m_i}(T_e - T_i), \qquad (4.17)$$

where

$$\nu_{ei} := \frac{16\sqrt{\pi}}{3}\frac{e^4 n_i \ln\Lambda}{(4\pi\epsilon_0)^2 m_e^2 v_{te}^3} \qquad (4.18)$$

is the electron-ion collision frequency as defined by Braginskii (Braginskii 1958). Note that Braginskii's definitions of ν_{ii} and ν_{ei} differ by a factor of $\sqrt{2}$.

The ion kinetic equation is

$$\frac{\partial F_i}{\partial t} + \dot{z}_i \frac{\partial F_i}{\partial z} + \dot{w}_{\|i}\frac{\partial F_i}{\partial w_\|} + \dot{w}_{\perp i}\frac{\partial F_i}{\partial w_\perp} = \dot{F}_i + \mathcal{C}_{ii} + \mathcal{C}_{in} + \mathcal{C}_{i,\mathrm{ion}} + \mathcal{S}_i. \qquad (4.19)$$

Here, we have defined the coefficients

$$\dot{z}_s[F_s, u_{s\|}, v_{ts}](z, w_\|, t) := u_{s\|} + v_{ts}w_\|, \qquad (4.20)$$

$$\dot{w}_{\|s}[F_s, n_s, u_{s\|}, v_{ts}](z, w_\|, t) := \frac{1}{n_s m_s v_{ts}}\frac{\partial p_{s\|}}{\partial z}$$

$$+ \frac{2w_\|}{3n_s m_s v_{ts}^2}\left[\frac{\partial q_{s\|}}{\partial z} + \left(p_{s\|} - \frac{3}{2}n_s m_s v_{ts}^2\right)\frac{\partial u_{s\|}}{\partial z}\right] - w_\|^2\frac{\partial v_{ts}}{\partial z}, \quad (4.21)$$

$$\dot{w}_{\perp s}[F_s, n_s, u_{s\|}, v_{ts}](z, w_\|, w_\perp, t) := \frac{2w_\perp}{3n_s m_s v_{ts}^2}\left(\frac{\partial q_{s\|}}{\partial z} + p_{s\|}\frac{\partial u_{s\|}}{\partial z}\right) - w_\| w_\perp\frac{\partial v_{ts}}{\partial z} \quad (4.22)$$

and

$$\dot{F}_s[F_s, n_s, u_{s\|}, v_{ts}](z, w_\|, w_\perp, t) := \left[w_\|\left(3\frac{\partial v_{ts}}{\partial z} - \frac{v_{ts}}{n_s}\frac{\partial n_s}{\partial z}\right)\right.$$

$$\left. - \frac{2}{n_s m_s v_{ts}^2}\left(\frac{\partial q_{s\|}}{\partial z} + \left(p_{s\|} - \frac{1}{2}n_s m_s v_{ts}^2\right)\frac{\partial u_{s\|}}{\partial z}\right)\right]F_s. \qquad (4.23)$$

We have also defined a modified source $\mathcal{S}_i$ and several modified collision operators. The modified source is given by

$$\mathcal{S}_s[S_s, F_s, n_s, u_{s\|}, v_{ts}](z, w_\|, w_\perp, t)$$

$$:= -\left[\frac{F_s}{n_s}\int S_s\,\mathrm{d}^3 v - \frac{v_{ts}^3}{n_s}S_s(z, u_{s\|} + v_{ts}w_\|, v_{ts}w_\perp, t)\right]$$

$$+ \frac{\partial}{\partial w_\|}\left[F_s\left(\frac{1}{n_s v_{ts}}\int(v_\| - u_{s\|})S_s\,\mathrm{d}^3 v + \frac{w_\|}{3n_s v_{ts}^2}\int\left(|\mathbf{v} - u_{s\|}\hat{\mathbf{z}}|^2 - \frac{3}{2}v_{ts}^2\right)S_s\,\mathrm{d}^3 v\right)\right]$$

$$+ \frac{1}{w_\perp}\frac{\partial}{\partial w_\perp}\left[\frac{w_\perp^2 F_s}{3n_s v_{ts}^2}\int\left(|\mathbf{v} - u_{s\|}\hat{\mathbf{z}}|^2 - \frac{3}{2}v_{ts}^2\right)S_s\,\mathrm{d}^3 v\right]. \qquad (4.24)$$

Note that the differential terms in this modified source could have been included in the definitions of the coefficients $\dot{w}_{\|i}$, $\dot{w}_{\perp i}$ and $\dot{F}_i$, but we have decided to make them part of a modified source instead to separate the effect of the source clearly. We will do the same for collisions. This split should not be taken as a suggestion on how to implement these terms in a code. The modified collisions operators are described in Appendix A.

The fluid equations for the neutrals are

$$\frac{\partial n_n}{\partial t} + \frac{\partial}{\partial z}\left(n_n u_{n\parallel}\right) = -n_n n_e R_{\text{ion}} + \int S_n \, \mathrm{d}^3 v, \qquad (4.25)$$

$$n_n m_i \left(\frac{\partial u_{n\parallel}}{\partial t} + u_{n\parallel}\frac{\partial u_{n\parallel}}{\partial z}\right) = -\frac{\partial p_{n\parallel}}{\partial z} + n_n m_i n_i R_{in}(u_{i\parallel} - u_{n\parallel})$$
$$+ \int m_i(v_\parallel - u_{n\parallel})S_n \, \mathrm{d}^3 v \qquad (4.26)$$

and

$$\frac{3}{2}n_n\left(\frac{\partial T_n}{\partial t} + u_{n\parallel}\frac{\partial T_n}{\partial z}\right) = -\frac{\partial q_{n\parallel}}{\partial z} - p_{n\parallel}\frac{\partial u_{n\parallel}}{\partial z} + \frac{3}{2}n_n n_i R_{in}(T_i - T_n)$$
$$+ \frac{1}{2}n_n m_i n_i R_{in}(u_{n\parallel} - u_{i\parallel})^2 + \int \frac{1}{2}m_i|\mathbf{v} - u_{n\parallel}\hat{\mathbf{z}}|^2 C_{ne} \, \mathrm{d}^3 v$$
$$+ \int \left(\frac{1}{2}m_i|\mathbf{v} - u_{n\parallel}\hat{\mathbf{z}}|^2 - \frac{3}{2}T_n\right)S_n \, \mathrm{d}^3 v. \qquad (4.27)$$

As with electron-ion collisions, the term $(1/2)\int m_i|\mathbf{v} - u_{n\parallel}\hat{\mathbf{z}}|^2 C_{ne} \, \mathrm{d}^3 v$ only becomes important when the collisions are sufficiently frequent that the distribution functions are close to Maxwellians, giving

$$\int \frac{1}{2}m_i|\mathbf{v} - u_{n\parallel}\hat{\mathbf{z}}|^2 C_{ne}[f_n, f_e] \, \mathrm{d}^3 v \simeq \int \frac{1}{2}m_i|\mathbf{v} - u_{n\parallel}\hat{\mathbf{z}}|^2 C_{ne}[f_{Mn}, f_{Me}] \, \mathrm{d}^3 v$$
$$\simeq \frac{3n_e m_e n_n R_{en}}{m_i}(T_e - T_n), \qquad (4.28)$$

The neutral kinetic equation is

$$\frac{\partial F_n}{\partial t} + \dot{z}_n\frac{\partial F_n}{\partial z} + \dot{w}_{\parallel n}\frac{\partial F_n}{\partial w_\parallel} + \dot{w}_{\perp n}\frac{\partial F_n}{\partial w_\perp} = \dot{F}_n + \mathcal{C}_{ni} + \mathcal{S}_n. \qquad (4.29)$$

The modified charge exchange collision operator $\mathcal{C}_{ni}$ is described in Appendix B.

Equations (4.19) and (4.29) for F_i and F_n are constructed such that conditions (4.7), (4.8) and (4.9) are satisfied at all times if they are satisfied at $t = 0$. In practice, this property has to be enforced in the numerical method. We have found an algorithm that works well and we have discussed it in report 2047357-TN-04-02 M2.2.

4.3. *Boundary conditions*

These equations for ions and neutrals have to be solved with the boundary conditions in equations (3.7) and (3.8). For n_s, $u_{s\parallel}$, v_{ts} and F_s known at time t, we can construct f_s at $z = 0$ and $z = L$, and we can apply boundary conditions (3.7) and (3.8). We can then use the resulting f_s to obtain n_s, $u_{s\parallel}$, v_{ts} and F_s, and to calculate $p_{s\parallel}$ and $q_{s\parallel}$, closing the system of equations.

We note that this is the only place where the full distribution function f_s is needed. Depending on the numerical method chosen to solve these equations, reconstructing f_s could be expensive. This is one of the problems that needs to be addressed in the proxy apps.

5. 1D moment drift kinetics for electrons

The equation for F_e can be significantly simplified using the expansion in $\sqrt{m_e/m_i}$. The electron source S_e is usually of order

$$S_e \sim \frac{v_{ti}f_e}{L} \tag{5.1}$$

because the ions control the dynamics due to their large mass, and the electrons adapt to the ions because of collisions and quasineutrality. Thus, S_e is a small term in equation (2.4) by a factor of $v_{ti}/v_{te} \sim \sqrt{m_e/m_i} \ll 1$. Moreover, the electron parallel flow $u_{e\parallel}$ is of the order of the ion parallel flow because boundary conditions (3.3) and (3.5) impose that these two velocities be equal at $z = 0$ and $z = L$, and quasineutrality keeps the difference of the order of v_{ti}. As a result, $u_{e\parallel} \sim u_{i\parallel} \sim v_{ti} \ll v_{te}$ and we can neglect $u_{e\parallel}$ to lowest order in most terms in equation (2.4) – the two exceptions in which $u_{e\parallel}$ cannot be neglected are given in Appendix C. Finally, the effect of ionization, modeled by $C_{e,\mathrm{ion}} \sim n_n R_{\mathrm{ion}} f_e$ is, according to equation (4.10), of the order of $f_e v_{ti}/L$ and thus also small.

5.1. *Electron kinetic equation*

Employing the expansion in $\sqrt{m_e/m_i} \ll 1$, the kinetic equation for electrons becomes

$$\dot{z}_e \frac{\partial F_e}{\partial z} + \dot{w}_{\parallel e} \frac{\partial F_e}{\partial w_{\parallel}} + \dot{w}_{\perp e} \frac{\partial F_e}{\partial w_{\perp}} = \dot{F}_e + \mathcal{C}_{ee} + \mathcal{C}_{ei} + \mathcal{C}_{en}, \tag{5.2}$$

where

$$\dot{z}_e[F_e, v_{te}](z, w_{\parallel}, t) := v_{te}w_{\parallel}, \tag{5.3}$$

$$\dot{w}_{\parallel e}[F_e, u_{e\parallel}, v_{te}](z, w_{\parallel}, t) := \frac{1}{n_e m_e v_{te}} \frac{\partial p_{e\parallel}}{\partial z} + \frac{2w_{\parallel}}{3n_e m_e v_{te}^2} \frac{\partial q_{e\parallel}}{\partial z} - w_{\parallel}^2 \frac{\partial v_{te}}{\partial z}, \tag{5.4}$$

$$\dot{w}_{\perp e}[F_e, u_{e\parallel}, v_{te}](z, w_{\parallel}, w_{\perp}, t) := \frac{2w_{\perp}}{3n_e m_e v_{te}^2} \frac{\partial q_{e\parallel}}{\partial z} - w_{\parallel}w_{\perp} \frac{\partial v_{te}}{\partial z} \tag{5.5}$$

and

$$\dot{F}_e[F_e, u_{e\parallel}, v_{te}](z, w_{\parallel}, w_{\perp}, t) := \left[w_{\parallel} \left(3\frac{\partial v_{te}}{\partial z} - \frac{v_{te}}{n_e} \frac{\partial n_e}{\partial z} \right) - \frac{2}{n_e m_e v_{te}^2} \frac{\partial q_{e\parallel}}{\partial z} \right] F_e. \tag{5.6}$$

The modified collision operators $\mathcal{C}_{ee}$, $\mathcal{C}_{ei}$ and $\mathcal{C}_{en}$ are described in Appendix C. Note that, in equations (C 1) and (C 6), we have kept small terms that scale with $(u_{s\parallel} - u_{e\parallel})/v_{te} \sim \sqrt{m_e/m_i} \ll 1$, where $s = i, n$. These terms are kept to ensure that we recover the Braginskii equations in the appropriate limit (see subsection 5.4).

We have ensured that equation (5.2) is compatible with conditions (4.7), (4.8) and (4.9) by keeping terms that are second order in $u_{e\parallel}$ in the collision operators in Appendix C. Indeed, multiplying equation (5.2) by 1, $w_{\parallel}$ and $w_{\parallel}^2 + w_{\perp}^2$ and integrating over velocities gives $0 = 0$. Despite the fact that we do not keep all possible terms that are second order in $\sqrt{m_e/m_i}$ in the collision operator, we will see in subsection 5.4 that proposed model recovers the regimes of interest.

Conditions (4.7), (4.8) and (4.9) have to be imposed on F_e when solving the kinetic equation (5.2). One possible way to impose these conditions is to include the term $\partial F_e/\partial t$ in equation (5.2) so that we can evolve F_e to a steady state solution. With this approach, if F_e satisfies conditions (4.7), (4.8) and (4.9) at $t = 0$, it will satisfy them at all times.

5.2. *Electron fluid equations*

Once we know F_e, we can calculate the fluid equations for electrons.

- The electron continuity equation is

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial z}\left(n_e u_{e\|}\right) = -n_e n_n R_{\text{ion}} + \int S_e \, d^3 v. \tag{5.7}$$

Subtracting this equation from equation (4.10) and using quasineutrality, we obtain the current conservation equation

$$\frac{\partial}{\partial z}\left[n_e\left(u_{i\|} - u_{e\|}\right)\right] = \int S_i \, d^3 v - \int S_e \, d^3 v. \tag{5.8}$$

This equation can be used to calculate $u_{e\|}$.

- The electron parallel momentum equation simplifies to

$$0 = -\frac{\partial p_{e\|}}{\partial z} + e n_e \frac{\partial \phi}{\partial z} + F_{ei\|} + n_e m_e n_n R_{en}(u_{n\|} - u_{e\|}), \tag{5.9}$$

where

$$
\begin{aligned}
&F_{ei\|}[F_e, n_e, n_i, u_{e\|}, u_{e\|}, v_{te}, v_{ti}](z,t) \\
&\quad := -\frac{8\pi^2 e^4 n_e n_i \ln \Lambda}{(4\pi\epsilon_0)^2 m_e v_{te}^2} \int_{-\infty}^{\infty} dw_\| \int_0^{\infty} dw_\perp \frac{w_\perp (w_\| - (u_{i\|} - u_{e\|})/v_{te}) F_e}{[(w_\| - (u_{i\|} - u_{e\|})/v_{te})^2 + w_\perp^2]^{3/2}}
\end{aligned} \tag{5.10}
$$

is the friction force between electrons and ions. Equation (5.9) can be used to calculate the potential ϕ. Since the potential at $\phi(0,t)$ is determined by equation (3.5), we can start integrating ϕ at $z = 0$. With the potential $\phi(L,t)$ that we obtain from this integration and condition (3.3), we can calculate the potential difference between the two walls, ϕ_w.

- The electron energy equation is

$$
\begin{aligned}
\frac{3}{2} n_e \left(\frac{\partial T_e}{\partial t} + u_{e\|}\frac{\partial T_e}{\partial z}\right) = {}& -\frac{\partial q_{e\|}}{\partial z} - p_{e\|}\frac{\partial u_{e\|}}{\partial z} + \int \frac{1}{2} m_e |\mathbf{v} - u_{e\|}\hat{\mathbf{z}}|^2 C_{e,\text{ion}} \, d^3 v \\
& + \int \frac{1}{2} m_e |\mathbf{v} - u_{e\|}\hat{\mathbf{z}}|^2 C_{ei}\left[1 + O\left(\frac{m_e}{m_i}\right)\right] d^3 v \\
& + \int \frac{1}{2} m_e |\mathbf{v} - u_{e\|}\hat{\mathbf{z}}|^2 C_{en}\left[1 + O\left(\frac{m_e}{m_i}\right)\right] d^3 v \\
& + \int \left(\frac{1}{2} m_e |\mathbf{v} - u_{e\|}\hat{\mathbf{z}}|^2 - \frac{3}{2} T_e\right) S_e \, d^3 v.
\end{aligned} \tag{5.11}
$$

For the integral over the ionization collision operator, we use the model

$$\int \frac{1}{2} m_e |\mathbf{v} - u_{e\|}\hat{\mathbf{z}}|^2 C_{e,\text{ion}} \, d^3 v = -n_e n_n R_{\text{ion}} E_{\text{ion}}, \tag{5.12}$$

where E_{ion} is the ionization energy cost that includes in it radiation from excited states. The integrals over C_{ei} and C_{en} are only sufficiently large when collisions are large. In this limit, all the species are Maxwellian and we can easily calculate the integrals over C_{ei} and C_{en} to higher order in the mass ratio expansion, finding

$$
\begin{aligned}
\int \frac{1}{2} m_e |\mathbf{v} - u_{e\|}\hat{\mathbf{z}}|^2 C_{ei}[f_e, f_i]\left[1 + O\left(\frac{m_e}{m_i}\right)\right] d^3 v \simeq {}& \frac{3 n_e m_e \nu_{ei}}{m_i}(T_i - T_e) \\
& + F_{ei\|}(u_{i\|} - u_{e\|})
\end{aligned} \tag{5.13}
$$

and

$$\int \frac{1}{2} m_e |\mathbf{v} - u_{e\|}\hat{\mathbf{z}}|^2 C_{en}[f_e, f_n] \left[1 + O\left(\frac{m_e}{m_i}\right)\right] \mathrm{d}^3 v \simeq \frac{3 n_e m_e n_n R_{en}}{m_i}(T_n - T_e)$$
$$+ n_e m_e n_n R_{en}(u_{n\|} - u_{e\|})^2. \quad (5.14)$$

5.3. *Boundary conditions*

These equations for electrons have to be solved with the boundary conditions in equations (3.1) and (3.4). As for ions and neutrals, the best way to impose these boundary conditions is to transform back to f_e, apply the boundary conditions and then calculate F_e, n_e, m_e, v_{te}, $p_{e\|}$ and $q_{e\|}$ from f_e.

5.4. *Some limits of interest*

We finish by showing that these equations recover the desired result in the two limits of interest: a modified Braginskii limit with neutrals, and a collisionless limit.

5.4.1. *Braginskii-like equations*

For the Braginskii-like limit, we use the orderings suggested in (Catto 1994; Helander *et al.* 1994): we assume that the ion-ion, ion-neutral, electron-electron, electron-ion and electron-neutral collisions are so frequent that their collision frequencies $\nu_{ss'}$ satisfy

$$\frac{\nu_{ss'} L}{v_{ts}} \sim \sqrt{\frac{m_i}{m_e}} \gg 1, \quad (5.15)$$

whereas the ionization frequencies $n_e R_{\mathrm{ion}}$ and $n_n R_{\mathrm{ion}}$ are of order v_{ti}/L.

In this limit, the charge exchange collisional terms dominate in the ion and neutral momentum and energy equations, forcing $u_{i\|} = u_{n\|}$ and $T_i = T_n$. We use $u_{h\|}$ and T_h to denote the average flow and temperature of the heavy species. By summing the ion and neutral momentum equations (4.11) and (4.26), we find the equation for $u_{h\|}$,

$$(n_i + n_n)m_i \left(\frac{\partial u_{h\|}}{\partial t} + u_{h\|}\frac{\partial u_{h\|}}{\partial z}\right) = -\frac{\partial}{\partial z}(p_{i\|} + p_{n\|}) - en_i\frac{\partial \phi}{\partial z}$$
$$+ \int m_i(v_\| - u_{h\|})(S_i + S_n)\,\mathrm{d}^3 v. \quad (5.16)$$

By summing the ion and neutral energy equations (4.12) and (4.27), we find the equation for T_h,

$$\frac{3}{2}(n_i + n_n)\left(\frac{\partial T_h}{\partial t} + u_{h\|}\frac{\partial T_h}{\partial z}\right) = -\frac{\partial}{\partial z}(q_{i\|} + q_{n\|}) - (p_{i\|} + p_{n\|})\frac{\partial u_{h\|}}{\partial z}$$
$$+ \frac{3 n_e m_e(\nu_{ei} + n_n R_{en})}{m_i}(T_e - T_h) + \int \left(\frac{1}{2}m_i|\mathbf{v} - u_{h\|}\hat{\mathbf{z}}|^2 - \frac{3}{2}T_h\right)(S_i + S_n)\,\mathrm{d}^3 v.$$
$$(5.17)$$

Note that in this energy equation, the terms due to electron-ion and electron-neutral collisions are of the same order as the other terms.

In the electron fluid equations, the collisional friction terms in the electron momentum equation (5.9) are comparable to the pressure and electric field terms. The electron heat flux term and the terms related to the electron-ion and electron-neutral collisions in the electron energy equation (5.11) are also comparable to the rest of the terms.

In the kinetic equations, the collisions dominate and lead to distribution functions that are Maxwellian to lowest order. We can use the kinetic equations to find the corrections

to the Maxwellian. For ions and neutrals, the corrections to the Maxwellian do not give large contributions to either the parallel pressure or the heat flux. For the electrons, however, the correction gives important contributions to the friction force in the momentum equation (5.9) and to the electron heat flux in the energy equation (5.11). If we write $F_e = F_M + F_{e1} + \ldots$, with $F_M := \pi^{-3/2} \exp(-w_\parallel^2 - w_\perp^2)$, we can find the equation for F_{e1} (Braginskii 1958),

$$
\mathcal{C}_{ee}^{(\ell)}[F_{e1}] + \mathcal{C}_{ei}^{(\ell)}\left[F_{e1} - \frac{2(u_{h\parallel} - u_{e\parallel})w_\parallel}{v_{te}}F_M\right] + \mathcal{C}_{en}^{(\ell)}\left[F_{e1} - \frac{2(u_{h\parallel} - u_{e\parallel})w_\parallel}{v_{te}}F_M\right]
$$
$$
+ \frac{16\pi^2 e^4 n_i \ln\Lambda}{(4\pi\epsilon_0)^2 m_e^2 v_{te}^3}w_\parallel F_M \int_{-\infty}^{\infty} \mathrm{d}w_\parallel' \int_0^\infty \mathrm{d}w_\perp' \, \frac{w_\perp' w_\parallel' F_{e1}(z, w_\parallel', w_\perp', t)}{(w_\parallel'^2 + w_\perp'^2)^{3/2}}
$$
$$
= \left[v_{te}w_\parallel\left(w_\parallel^2 + w_\perp^2 - \frac{5}{2}\right)\frac{\partial}{\partial z}\ln T_e + \frac{32\sqrt{\pi}e^4 n_i \ln\Lambda (u_{h\parallel} - u_{e\parallel})w_\parallel}{3(4\pi\epsilon_0)^2 m_e^2 v_{te}^4}\right.
$$
$$
\left. + \frac{2n_n R_{en}(u_{h\parallel} - u_{e\parallel})w_\parallel}{v_{te}}\right]F_M. \quad (5.18)
$$

Here we have neglected the electron heat flux $q_{e\parallel}$ because it is proportional to F_{e1}. The collision operators $\mathcal{C}_{ee}^{(\ell)}$, $\mathcal{C}_{ei}^{(\ell)}$ and $\mathcal{C}_{en}^{(\ell)}$ are the linearized collision operators, given in Appendix D. Note that the terms proportional to $u_{h\parallel} - u_{e\parallel}$, needed to recover the friction force and electron heat flux in Braginskii (1958) (see subsection 5.4), come from the electron-ion and electron-electron collision operators in equations (C 1) and (C 6).

We finish by pointing out that F_e satisfies conditions (4.7), (4.8) and (4.9) to the order that we have calculated it. Since F_M satisfies these conditions, the conditions for F_{e1} become

$$
2\pi \int_{-\infty}^{\infty} \mathrm{d}w_\parallel \int_0^\infty \mathrm{d}w_\perp \, w_\perp F_{e1}(z, w_\parallel, w_\perp, t) = 0, \quad (5.19)
$$

$$
2\pi \int_{-\infty}^{\infty} \mathrm{d}w_\parallel \int_0^\infty \mathrm{d}w_\perp \, w_\perp w_\parallel F_{e1}(z, w_\parallel, w_\perp, t) = 0 \quad (5.20)
$$

and

$$
2\pi \int_{-\infty}^{\infty} \mathrm{d}w_\parallel \int_0^\infty \mathrm{d}w_\perp \, w_\perp (w_\parallel^2 + w_\perp^2)F_{e1}(z, w_\parallel, w_\perp, t) = 0. \quad (5.21)
$$

These conditions determine the pieces of F_{e1} that are in the kernel of the operators in the left side of equation (5.18).

5.4.2. *Collisionless electron equations*

We call this limit collisionless in contraposition to the collisional Braginsikii-like limit discussed above, but we still keep collisions. We assume that ion-ion, ion-neutral, electron-electron, electron-ion, electron-neutral and ion-neutral collisions satisfy

$$
\frac{\nu_{ss'}L}{v_{ts}} \sim 1. \quad (5.22)
$$

The ionization frequencies $n_e R_{\mathrm{ion}}$ and $n_n R_{\mathrm{ion}}$ are still assumed to be of order v_{ti}/L.

In this limit, we can neglect the collisional coupling between the heavy species (ions and neutrals) and electrons in the fluid equations. In the electron energy equation (5.11), the dominant term is $\partial q_{e\parallel}/\partial z$, giving

$$
\frac{\partial q_{e\parallel}}{\partial z} \simeq 0 \quad (5.23)
$$

In the electron momentum equation (5.9), the friction forces are negligible, giving

$$\frac{\partial p_{e\parallel}}{\partial z} \simeq e n_e \frac{\partial \phi}{\partial z}. \tag{5.24}$$

Using these results in the electron kinetic equation (5.2) and neglecting $u_{i\parallel} - u_{e\parallel}$ and $u_{n\parallel} - u_{e\parallel}$ in the collision operators in equation (C 1) and (C 6), we find

$$\left[v_{te} w_\parallel \frac{\partial}{\partial z} + \left(\frac{e}{m_e v_{te}} \frac{\partial \phi}{\partial z} - w_\parallel^2 \frac{\partial v_{te}}{\partial z} \right) \frac{\partial}{\partial w_\parallel} - w_\parallel w_\perp \frac{\partial v_{te}}{\partial z} \frac{\partial}{\partial w_\perp} \right] \left(\frac{n_e F_e}{v_{te}^3} \right)$$
$$= \frac{n_e}{v_{te}^3} \left(\mathcal{C}_{ee} + \mathcal{C}_{ei}^{(\ell)} + \mathcal{C}_{en}^{(\ell)} \right). \tag{5.25}$$

The collision operators $\mathcal{C}_{ei}^{(\ell)}$ and $\mathcal{C}_{en}^{(\ell)}$ are defined in Appendix D.

We can use entropy production of the electron-electron, electron-ion and electron-neutral collision operators to solve equation (5.25). We multiply equation (5.25) by $-\ln(n_e F_e / v_{te}^3)$ to find

$$\left[v_{te} w_\parallel \frac{\partial}{\partial z} + \left(\frac{e}{m_e v_{te}} \frac{\partial \phi}{\partial z} - w_\parallel^2 \frac{\partial v_{te}}{\partial z} \right) \frac{\partial}{\partial w_\parallel} - w_\parallel w_\perp \frac{\partial v_{te}}{\partial z} \frac{\partial}{\partial w_\perp} \right] \left[\frac{n_e F_e}{v_{te}^3} \left(1 - \ln \left(\frac{n_e F_e}{v_{te}^3} \right) \right) \right]$$
$$= -\frac{n_e}{v_{te}^3} \ln \left(\frac{n_e F_e}{v_{te}^3} \right) \left(\mathcal{C}_{ee} + \mathcal{C}_{ei}^{(\ell)} + \mathcal{C}_{en}^{(\ell)} \right). \tag{5.26}$$

Multiplying by v_{te}^3 and integrating over velocity space, we obtain

$$\frac{\partial}{\partial z} \left[n_e v_{te} \int \left(1 - \ln \left(\frac{n_e F_e}{v_{te}^3} \right) \right) F_e w_\parallel w_\perp \, \mathrm{d}w_\parallel \, \mathrm{d}w_\perp \right]$$
$$= -n_e \int \ln \left(\frac{n_e F_e}{v_{te}^3} \right) \left(\mathcal{C}_{ee} + \mathcal{C}_{ei}^{(\ell)} + \mathcal{C}_{en}^{(\ell)} \right) w_\perp \, \mathrm{d}w_\parallel \, \mathrm{d}w_\perp. \tag{5.27}$$

Integrating equation (5.27) over z gives

$$\left[n_e v_{te} \int \left(1 - \ln \left(\frac{n_e F_e}{v_{te}^3} \right) \right) F_e w_\parallel w_\perp \, \mathrm{d}w_\parallel \, \mathrm{d}w_\perp \right]_{z=0}^{z=L}$$
$$= -n_e \int_0^L \mathrm{d}z \int \ln \left(\frac{n_e F_e}{v_{te}^3} \right) \left(\mathcal{C}_{ee} + \mathcal{C}_{ei}^{(\ell)} + \mathcal{C}_{en}^{(\ell)} \right) w_\perp \, \mathrm{d}w_\parallel \, \mathrm{d}w_\perp. \tag{5.28}$$

Conditions (3.3) and (3.5) impose that only a few electrons leave the system towards the wall. The number of electrons that leave is small by a factor of $\sqrt{m_e/m_i} \ll 1$ and thus, the left side of equation (5.28) can be neglected, finally giving

$$-\int_0^L \mathrm{d}z \int \ln \left(\frac{n_e F_e}{v_{te}^3} \right) \left(\mathcal{C}_{ee} + \mathcal{C}_{ei}^{(\ell)} + \mathcal{C}_{en}^{(\ell)} \right) w_\perp \, \mathrm{d}w_\parallel \, \mathrm{d}w_\perp \simeq 0. \tag{5.29}$$

The integrand under the integral over z is the entropy production, and it is positive unless the distribution function F_e is the Maxwellian F_M. Hence, F_e is a Maxwellian to lowest order in the expansion in $\sqrt{m_e/m_i}$.

Substituting the Maxwellian into equation (5.25), we find that the right side of the equation vanishes. For the left side of the equation to be zero for all $w_\parallel$ and $w_\perp$, we need $\partial v_{te}/\partial z = 0$ and

$$n_e(z,t) = N_e(t) \exp \left(\frac{e \phi(z,t)}{T_e(t)} \right). \tag{5.30}$$

The value of $N_e(t)$ is calculated from quasineutrality,

$$N_e(t) \int_0^L \exp\left(\frac{e\phi(z,t)}{T_e(t)}\right) \,\mathrm{d}z = \int_0^L n_i(z,t)\,\mathrm{d}z. \tag{5.31}$$

The value of $T_e(t)$ is obtained from the electron energy equation. Integrating the energy equation (5.11) over z, we find

$$\frac{3}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[N_e T_e \int_0^L \exp\left(\frac{e\phi}{T_e}\right)\mathrm{d}z\right] = -\left[\frac{5}{2}n_e u_{e\parallel}T_e + q_{e\parallel}\right]_{z=0}^{z=L}$$

$$+ \int_0^L \left(e n_e \frac{\partial\phi}{\partial z} - n_e n_n R_{\mathrm{ion}} E_{\mathrm{ion}} + \int \frac{1}{2} m_e v^2 S_e \,\mathrm{d}^3 v\right)\mathrm{d}z. \tag{5.32}$$

It might seem that the heat flux should vanish here because the distribution function is a Maxwellian to lowest order. In reality, the real F_e is sufficiently far from a Maxwellian at large velocities to induce a significant heat flux. Indeed, to satisfy boundary conditions (3.1) and (3.4), the distribution function must vanish exactly in certain regions of phase space. Since these regions are at large velocities, the techniques used to calculate the collisional losses into the loss cones of mirror machines can be used (Pastukhov 1974).

6. Discussion

The model that we propose is comprised of:
- the three fluid equations (4.10), (4.11) and (4.12) for ions that have to be solved in conjunction with the ion kinetic equation (4.19);
- the three fluid equations (4.25), (4.26) and (4.27) for neutrals that have to be solved in conjunction with the neutral kinetic equation (4.29);
- the two fluid equations (5.8) and (5.11) for electrons that have to be solved in conjunction with the electron kinetic equation (5.2) with the modified coefficients (5.3), (5.4), (5.5) and (5.6); and
- the electron parallel momentum equation (5.9) for the potential.

The boundary conditions for this system of equations are described in section 3.

To test the model proposed in this report, we will first extend the existing code based on adiabatic electrons, which we prove to a be a good approximation for collisionless plasmas in section 5.4, to wall boundary conditions. We will then explore the effect of adding electrons. For most physics of interest, it is sufficient to use simplified ion-ion and electron-electron collision operators, and for this reason we do not expect to implement a full Fokker-Planck collision operator.

Appendix A. Modified collision operators for the ion kinetic equation

The modified Fokker-Planck like-particle collision operator is

$$\mathcal{C}_{ss}[F_s, n_s, v_{ts}](z, w_\parallel, w_\perp, t)$$

$$:= \frac{2\pi e^4 n_s \ln\Lambda}{(4\pi\epsilon_0)^2 m_s^2 v_{ts}^3}\left\{\frac{\partial}{\partial w_\parallel}\left(\mathcal{D}_{\parallel\parallel}[F_s]\frac{\partial F_s}{\partial w_\parallel} + \mathcal{D}_{\parallel\perp}[F_s]\frac{\partial F_s}{\partial w_\perp} + \mathcal{P}_\parallel[F_s]F_s\right)\right.$$

$$\left. + \frac{1}{w_\perp}\frac{\partial}{\partial w_\perp}\left[w_\perp\left(\mathcal{D}_{\parallel\perp}[F_s]\frac{\partial F_s}{\partial w_\parallel} + \mathcal{D}_{\perp\perp}[F_s]\frac{\partial F_s}{\partial w_\perp} + \mathcal{P}_\perp[F_s]F_s\right)\right]\right\}. \tag{A1}$$

The coefficients needed for this collision operator are

$$
\mathcal{D}_{\|\|}[F_s](z, w_\|, w_\perp, t) := 4 \int_{-\infty}^{\infty} dw_\|' \int_0^\infty dw_\perp' \frac{w_\perp'}{\sqrt{(w_\| - w_\|')^2 + (w_\perp + w_\perp')^2}}
$$
$$
\times \left(K(\kappa) - \frac{(w_\| - w_\|')^2 E(\kappa)}{(w_\| - w_\|')^2 + (w_\perp - w_\perp')^2} \right) F_s(z, w_\|', w_\perp', t), \qquad (A\,2)
$$

$$
\mathcal{D}_{\|\perp}[F_s](z, w_\|, w_\perp, t) := 2 \int_{-\infty}^{\infty} dw_\|' \int_0^\infty dw_\perp' \frac{w_\perp'(w_\| - w_\|')}{w_\perp \sqrt{(w_\| - w_\|')^2 + (w_\perp + w_\perp')^2}}
$$
$$
\times \left(\frac{[(w_\| - w_\|')^2 - w_\perp^2 + w_\perp'^2] E(\kappa)}{(w_\| - w_\|')^2 + (w_\perp - w_\perp')^2} - K(\kappa) \right) F_s(z, w_\|', w_\perp', t), \quad (A\,3)
$$

$$
\mathcal{D}_{\perp\perp}[F_s](z, w_\|, w_\perp, t) := 2 \int_{-\infty}^{\infty} dw_\|' \int_0^\infty dw_\perp' \frac{w_\perp'}{w_\perp^2 \sqrt{(w_\| - w_\|')^2 + (w_\perp + w_\perp')^2}}
$$
$$
\times \left\{ 2w_\perp \left[\frac{w_\perp(w_\| - w_\|')^2}{(w_\| - w_\|')^2 + (w_\perp - w_\perp)^2} - w_\perp' \right] E(\kappa) \right.
$$
$$
\left. + [(w_\| - w_\|')^2 + w_\perp^2 + w_\perp'^2][K(\kappa) - E(\kappa)] \right\} F_s(z, w_\|', w_\perp', t), \quad (A\,4)
$$

$$
\mathcal{P}_\|[F_s](z, w_\|, w_\perp, t) := 8 \int_{-\infty}^{\infty} dw_\|' \int_0^\infty dw_\perp' \frac{w_\perp'(w_\| - w_\|')}{\sqrt{(w_\| - w_\|')^2 + (w_\perp + w_\perp')^2}}
$$
$$
\times \left(\frac{K(\kappa) - E(\kappa)}{(w_\| - w_\|')^2 + (w_\perp + w_\perp')^2} - \frac{E(\kappa)}{(w_\| - w_\|')^2 + (w_\perp - w_\perp')^2} \right) F_s(z, w_\|', w_\perp', t)
$$
$$
(A\,5)
$$

and

$$
\mathcal{P}_\perp[F_s](z, w_\|, w_\perp, t) := 4 \int_{-\infty}^{\infty} dw_\|' \int_0^\infty dw_\perp' \frac{w_\perp'}{w_\perp \sqrt{(w_\| - w_\|')^2 + (w_\perp + w_\perp')^2}}
$$
$$
\times \left(\frac{[(w_\| - w_\|')^2 - w_\perp^2 + w_\perp'^2] E(\kappa)}{(w_\| - w_\|')^2 + (w_\perp - w_\perp')^2} - K(\kappa) \right) F_s(z, w_\|', w_\perp', t). \qquad (A\,6)
$$

Here, $K(\kappa) := \int_0^{\pi/2} (1 - \kappa^2 \sin^2 \alpha)^{-1/2} \, d\alpha$ and $E(\kappa) := \int_0^{\pi/2} (1 - \kappa^2 \sin^2 \alpha)^{1/2} \, d\alpha$ are the elliptic integrals, and the function κ is

$$
\kappa(w_\|, w_\perp, w_\|', w_\perp') := \sqrt{\frac{4 w_\perp w_\perp'}{(w_\| - w_\|')^2 + (w_\perp + w_\perp')^2}}. \qquad (A\,7)
$$

The modified charge exchange collision operator for the ion kinetic equation is

$$\mathcal{C}_{in}[F_i, F_n, n_n, u_{i\parallel}, u_{n\parallel}, v_{ti}, v_{tn}](z, w_\parallel, w_\perp, t)$$

$$:= -n_n R_{in} \left[F_i - \frac{v_{ti}^3}{v_{tn}^3} F_n \left(z, \frac{u_{i\parallel} - u_{n\parallel}}{v_{tn}} + \frac{v_{ti}}{v_{tn}} w_\parallel, \frac{v_{ti}}{v_{tn}} w_\perp, t \right) \right]$$

$$+ n_n R_{in} \frac{\partial}{\partial w_\parallel} \left[\left(\frac{u_{n\parallel} - u_{i\parallel}}{v_{ti}} + \frac{w_\parallel}{2} \left(\frac{v_{tn}^2}{v_{ti}^2} - 1 + \frac{2(u_{n\parallel} - u_{i\parallel})^2}{3v_{ti}^2} \right) \right) F_i \right]$$

$$+ \frac{n_n R_{in}}{w_\perp} \frac{\partial}{\partial w_\perp} \left[\frac{w_\perp^2}{2} \left(\frac{v_{tn}^2}{v_{ti}^2} - 1 + \frac{2(u_{n\parallel} - u_{i\parallel})^2}{3v_{ti}^2} \right) F_i \right]. \tag{A 8}$$

Finally, the modified ionization collision operator for the ion kinetic equation is

$$\mathcal{C}_{i,\mathrm{ion}}[F_n, n_e, u_{i\parallel}, u_{n\parallel}, v_{ti}, v_{tn}](z, w_\parallel, w_\perp, t)$$

$$:= -n_e R_{\mathrm{ion}} \left[F_i - \frac{v_{ti}^3}{v_{tn}^3} F_n \left(z, \frac{u_{i\parallel} - u_{n\parallel}}{v_{tn}} + \frac{v_{ti}}{v_{tn}} w_\parallel, \frac{v_{ti}}{v_{tn}} w_\perp, t \right) \right]$$

$$+ n_e R_{\mathrm{ion}} \frac{\partial}{\partial w_\parallel} \left[\left(\frac{u_{n\parallel} - u_{i\parallel}}{v_{ti}} + \frac{w_\parallel}{2} \left(\frac{v_{tn}^2}{v_{ti}^2} - 1 + \frac{2(u_{n\parallel} - u_{i\parallel})^2}{3v_{ti}^2} \right) \right) F_i \right]$$

$$+ \frac{n_e R_{\mathrm{ion}}}{w_\perp} \frac{\partial}{\partial w_\perp} \left[\frac{w_\perp^2}{2} \left(\frac{v_{tn}^2}{v_{ti}^2} - 1 + \frac{2(u_{n\parallel} - u_{i\parallel})^2}{3v_{ti}^2} \right) F_i \right]. \tag{A 9}$$

Appendix B. Modified collision operators for the neutral kinetic equation

The modified charge exchange collision operator for the neutral kinetic equation is

$$\mathcal{C}_{ni}[F_n, F_i, n_i, u_{n\parallel}, u_{i\parallel}, v_{tn}, v_{ti}](z, w_\parallel, w_\perp, t)$$

$$:= -n_i R_{in} \left[F_n - \frac{v_{tn}^3}{v_{ti}^3} F_i \left(z, \frac{u_{n\parallel} - u_{i\parallel}}{v_{ti}} + \frac{v_{tn}}{v_{ti}} w_\parallel, \frac{v_{tn}}{v_{ti}} w_\perp, t \right) \right]$$

$$+ n_i R_{in} \frac{\partial}{\partial w_\parallel} \left[\left(\frac{u_{i\parallel} - u_{n\parallel}}{v_{tn}} + \frac{w_\parallel}{2} \left(\frac{v_{ti}^2}{v_{tn}^2} - 1 + \frac{2(u_{n\parallel} - u_{i\parallel})^2}{3v_{tn}^2} \right) \right) F_n \right]$$

$$+ \frac{n_i R_{in}}{w_\perp} \frac{\partial}{\partial w_\perp} \left[\frac{w_\perp^2}{2} \left(\frac{v_{ti}^2}{v_{tn}^2} - 1 + \frac{2(u_{n\parallel} - u_{i\parallel})^2}{3v_{tn}^2} \right) F_n \right]. \tag{B 1}$$

Appendix C. Modified collision operators for the electron kinetic equation

The electron-electron collision operator is described in equation (A 1).
The modified ion-electron collision operator is

$$\mathcal{C}_{ei}[F_e, n_i, u_{i\parallel}, u_{e\parallel}, v_{te}](z, w_\parallel, w_\perp, t)$$

$$:= \frac{2\pi e^4 n_i \ln \Lambda}{(4\pi\epsilon_0)^2 m_e^2 v_{te}^3} \left\{ \frac{\partial}{\partial w_\parallel} \left[\mathcal{M}_{\parallel\parallel} \frac{\partial F_e}{\partial w_\parallel} + \mathcal{M}_{\parallel\perp} \frac{\partial F_e}{\partial w_\perp} + \left(1 + \frac{2(u_{i\parallel} - u_{e\parallel})w_\parallel}{3v_{te}} \right) \mathcal{F}_\parallel F_e \right] \right.$$

$$\left. + \frac{1}{w_\perp} \frac{\partial}{\partial w_\perp} \left[w_\perp \left(\mathcal{M}_{\parallel\perp} \frac{\partial F_e}{\partial w_\parallel} + \mathcal{M}_{\perp\perp} \frac{\partial F_e}{\partial w_\perp} + \frac{2(u_{i\parallel} - u_{e\parallel})w_\perp}{3v_{te}} \mathcal{F}_\parallel F_e \right) \right] \right\}, \tag{C 1}$$

where

$$\mathcal{M}_{\|\|}[u_{e\|}, v_{te}, u_{i\|}](z, w_{\|}, w_{\perp}, t) := \frac{w_{\perp}^2}{[(w_{\|} - (u_{i\|} - u_{e\|})/v_{te})^2 + w_{\perp}^2]^{3/2}}, \qquad (C\,2)$$

$$\mathcal{M}_{\|\perp}[u_{e\|}, v_{te}, u_{i\|}](z, w_{\|}, w_{\perp}, t) := -\frac{(w_{\|} - (u_{i\|} - u_{e\|})/v_{te})^2 w_{\perp}}{[(w_{\|} - (u_{i\|} - u_{e\|})/v_{te})^2 + w_{\perp}^2]^{3/2}}, \qquad (C\,3)$$

$$\mathcal{M}_{\perp\perp}[u_{e\|}, v_{te}, u_{i\|}](z, w_{\|}, w_{\perp}, t) := \frac{(w_{\|} - (u_{i\|} - u_{e\|})/v_{te})^2}{[(w_{\|} - (u_{i\|} - u_{e\|})/v_{te})^2 + w_{\perp}^2]^{3/2}} \qquad (C\,4)$$

and

$$\mathcal{F}_{\|}[F_e, u_{e\|}, v_{te}, u_{i\|}](z, t)$$
$$:= -4\pi \int_{-\infty}^{\infty} \mathrm{d}w_{\|} \int_0^{\infty} \mathrm{d}w_{\perp} \frac{w_{\perp}[w_{\|} - (u_{i\|} - u_{e\|})/v_{te}]F_e(z, w_{\|}, w_{\perp}, t)}{[(w_{\|} - (u_{i\|} - u_{e\|})/v_{te})^2 + w_{\perp}^2]^{3/2}}. \qquad (C\,5)$$

The modified electron-neutral collision operator is

$$\mathcal{C}_{en}[F_e, n_n, u_{n\|}, u_{e\|}, v_{te}](z, w_{\|}, w_{\perp}, t) := -n_n R_{en} \left[F_e - \frac{1}{2} \int_0^{\pi} \sin\chi F_e(z, \overline{w}_{\|}, \overline{w}_{\perp}, t) \, \mathrm{d}\chi \right]$$
$$+ \frac{n_n R_{en}(u_{n\|} - u_{e\|})}{v_{te}} \frac{\partial}{\partial w_{\|}} \left[\left(1 + \frac{2(u_{n\|} - u_{e\|})w_{\|}}{3v_{te}} \right) F_e \right]$$
$$+ \frac{2n_n R_{en}(u_{n\|} - u_{e\|})^2}{3v_{te}^2 w_{\perp}} \frac{\partial}{\partial w_{\perp}} \left(w_{\perp}^2 F_e \right), \qquad (C\,6)$$

where

$$\overline{w}_{\|}[u_{e\|}, v_{te}, u_{n\|}](\chi, z, w_{\|}, w_{\perp}, t) := \frac{u_{n\|} - u_{e\|}}{v_{te}} + \cos\chi \sqrt{\left(w_{\|} - \frac{u_{n\|} - u_{e\|}}{v_{te}} \right)^2 + w_{\perp}^2} \quad (C\,7)$$

and

$$\overline{w}_{\perp}[u_{e\|}, v_{te}, u_{n\|}](\chi, z, w_{\|}, w_{\perp}, t) := \sin\chi \sqrt{\left(w_{\|} - \frac{u_{n\|} - u_{e\|}}{v_{te}} \right)^2 + w_{\perp}^2}. \qquad (C\,8)$$

Appendix D. Linearized collision operators for electrons

The linearized electron-electron collision operator is given by

$$\mathcal{C}_{ee}^{(\ell)}[F_{e1}, n_e, v_{te}](z, w_{\|}, w_{\perp}, t)$$
$$:= \frac{2\pi e^4 n_e \ln\Lambda}{(4\pi\epsilon_0)^2 m_e^2 v_{te}^3} \left\{ \frac{\partial}{\partial w_{\|}} \left(\mathcal{D}_{\|\|}[F_M] \frac{\partial F_{e1}}{\partial w_{\|}} + \mathcal{D}_{\|\perp}[F_M] \frac{\partial F_{e1}}{\partial w_{\perp}} + \mathcal{P}_{\|}[F_M] F_{e1} \right. \right.$$
$$\left. - 2w_{\|}\mathcal{D}_{\|\|}[F_{e1}] F_M - 2w_{\perp}\mathcal{D}_{\|\perp}[F_{e1}] F_M + \mathcal{P}_{\|}[F_{e1}] F_M \right)$$
$$+ \frac{1}{w_{\perp}} \frac{\partial}{\partial w_{\perp}} \left[w_{\perp} \left(\mathcal{D}_{\|\perp}[F_M] \frac{\partial F_{e1}}{\partial w_{\|}} + \mathcal{D}_{\perp\perp}[F_M] \frac{\partial F_{e1}}{\partial w_{\perp}} + \mathcal{P}_{\perp}[F_M] F_{e1} \right. \right.$$
$$\left. \left. \left. - 2w_{\|}\mathcal{D}_{\|\perp}[F_{e1}] F_M - 2w_{\perp}\mathcal{D}_{\perp\perp}[F_{e1}] F_M + \mathcal{P}_{\perp}[F_{e1}] F_M \right) \right] \right\}. \qquad (D\,1)$$

The coefficients are defined in Appendix A.

The linearized electron-ion collision operator is

$$\mathcal{C}_{ei}^{(\ell)}[F_{e1}, n_i, v_{te}](z, w_\parallel, w_\perp, t)$$
$$:= \frac{2\pi e^4 n_i \ln\Lambda}{(4\pi\epsilon_0)^2 m_e^2 v_{te}^3} \left\{ \frac{\partial}{\partial w_\parallel} \left[\frac{w_\perp^2}{(w_\parallel^2 + w_\perp^2)^{3/2}} \frac{\partial F_{e1}}{\partial w_\parallel} - \frac{w_\parallel w_\perp}{(w_\parallel^2 + w_\perp^2)^{3/2}} \frac{\partial F_{e1}}{\partial w_\perp} \right] \right.$$
$$\left. + \frac{1}{w_\perp} \frac{\partial}{\partial w_\perp} \left[w_\perp \left(-\frac{w_\parallel w_\perp}{(w_\parallel^2 + w_\perp^2)^{3/2}} \frac{\partial F_{e1}}{\partial w_\parallel} + \frac{w_\parallel^2}{(w_\parallel^2 + w_\perp^2)^{3/2}} \frac{\partial F_{e1}}{\partial w_\perp} \right) \right] \right\}. \quad \text{(D 2)}$$

Finally, the linearized electron-neutral collision operator is

$$\mathcal{C}_{en}^{(\ell)}[F_{e1}, n_n, v_{te}](z, w_\parallel, w_\perp, t) := -n_n R_{en} \left[F_{e1} \right.$$
$$\left. -\frac{1}{2} \int_0^\pi \sin\chi F_{e1}\left(z, \cos\chi\sqrt{w_\parallel^2 + w_\perp^2}, \sin\chi\sqrt{w_\parallel^2 + w_\perp^2}, t\right) \, \mathrm{d}\chi \right]. \quad \text{(D 3)}$$

REFERENCES

BRAGINSKII, S.I. 1958 Transport phenomena in a completely ionized two-temperature plasma. *Sov. Phys. JETP* **6**, 358.

CATTO, P.J. 1994 A short mean-free path, coupled neutral-ion transport description of a tokamak edge plasma. *Phys. Plasmas* **1**, 1936.

CHODURA, R. 1982 Plasma-wall transition in an oblique magnetic field. *Phys. Fluids* **25**, 1628.

CONNOR, J.W. 1977 An analytic solution for the distribution function of neutral particles in a Maxwellian plasma using the method of singular eigenfunctions. *Plasma Phys.* **19**, 853.

GERALDINI, A. 2021 Large gyro-orbit model of ion velocity distribution in plasma near a wall in a grazing-angle magnetic field. *J. Plasma Phys.* **87**, 905870113.

GERALDINI, A., PARRA, F.I. & MILITELLO, F. 2017 Gyrokinetic treatment of a grazing angle magnetic presheath. *Plasma Phys. Control. Fusion* **59**, 025015.

GERALDINI, A., PARRA, F.I. & MILITELLO, F. 2018 Solution to a collisionless shallow-angle magnetic presheath with kinetic ions. *Plasma Phys. Control. Fusion* **60**, 125002.

GERALDINI, A., PARRA, F.I. & MILITELLO, F. 2019 Dependence on ion temperature of shallow-angle magnetic presheaths with adiabatic electrons. *J. Plasma Phys.* **85**, 795850601.

HAZELTINE, R.D. 1973 Recursive derivation of drift-kinetic equation. *Plasma Phys.* **15**, 77–80.

HAZELTINE, R.D., CALVIN, M.D., VALANJU, P.M. & SOLANO, E.R. 1992 Analytical calculation of neutral transport and its effect on ions. *Nucl. Fusion* **32**, 3.

HELANDER, P., KRASHENINNIKOV, S.I. & CATTO, P.J. 1994 Fluid equations for a partially ionized plasma. *Phys. Plasmas* **1**, 3174.

KNUDSEN, M. 1916 Das Cosinusgesetz in der kinetischen Gastheorie. *Annal. Phys.* **353**, 1113.

PARKER, S.E., PROCASSINI, R.J. & BIRDSALL, C.K. 1993 A Suitable Boundary Condition for Bounded Plasma Simulation without Sheath Resolution. *J. Comput. Phys.* **104**, 41.

PASTUKHOV, V.P. 1974 Collisional losses of electrons from an adiabatic trap in a plasma with a positive potential. *Nucl. Fusion* **14**, 3.

ROSENBLUTH, M.N., MACDONALD, W.M. & JUDD, D.L. 1957 Fokker-Planck Equation for an Inverse-Square Force. *Phys. Rev.* **107**, 1.