Numerical implementation of a fluid model for electrons in a drift kinetic code

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1. Introduction

Calculation of the electrostatic potential is a challenge within the drift kinetic approximation. This is because, without further manipulation of the drift kinetic system of equations, there is no explicit equation to be solved for the potential: Instead, it appears as a parameter in the drift kinetic equations for both electrons and ions, which are then related to one another via quasineutrality. In report 2047357-TN-07-01 [1], we showed how the electrostatic potential could be calculated self-consistently for a 2D drift kinetic model with a helical magnetic field and with wall boundary conditions appropriate for open field lines. Considerable care had to be taken to obtain an expression for the electrostatic potential. In particular, a set of fluid equations had to be formulated for the electron dynamics, with closure via coupling to the kinetic equation for a modified electron particle distribution function. In this moment-kinetic formulation, the parallel electron momentum equation becomes the equation for the electrostatic potential, with an additional complication that the boundary value for the potential must be determined implicitly as it enters in the limits of integration for the parallel current at the plasma boundary.

Thus far we have side-stepped the issue of calculating the potential in our ProxyApps by using a Boltzmann response for the electron dynamics so that no electron equations need to be solved at all. We are now seeking to address this limitation, as we describe in this report. In particular, we have implemented the electron fluid equations for the 1+1D system described in report 2047357-TN-05-01 [2], with a Braginskii fluid closure for the electrons [3]. The main advantage of restricting our attention to the 1+1D case is the fact that in such a system there is no need to solve for the electrostatic

potential itself: Instead, we need only compute the parallel electric field, and this allows us to avoid for the moment the issue of how to solve for the electrostatic potential at the boundary of the domain.

To begin we will provide a brief overview of the physical system we aim to model, and provide the system of evolution equations for the various plasma species. We then demonstrate that the implementation appears to be working by providing numerical results and comparing with a test case.

2. Drift kinetic system of equations

We consider a plasma consisting of a single ion species of charge e and mass m_i, a single neutral species with mass $m_n = m_i$, and electrons with charge $-e$ and mass m_e. The plasma is immersed in a straight, homogeneous magnetic field of the form

$$\mathbf{B} = B\hat{\mathbf{z}}, \tag{1}$$

where z is the field-aligned coordinate, and $\hat{\mathbf{z}}$ is the unit vector in the direction of ∇z. We assume that the plasma is electrostatic and that the magnetic field terminates on each end at a conducting wall. The spatial domain we consider is $z \in [z_-, z_+]$, with $z_+ - z_- = L_z$. The boundaries of the domain in z are assumed to be the entrances to the magnetic pre-sheath.

We restrict our attention to parallel dynamics only and define the marginalised distribution function

$$f_s(z, v_\parallel, t) \doteq \left\langle \hat{f}_s(z, v_\parallel, v_\perp, \vartheta, t) \right\rangle \doteq 2\pi \int_0^\infty dv_\perp v_\perp \hat{f}_s, \tag{2}$$

with $\hat{f}_s$ the particle distribution function for species s, t the time, $v_\parallel$ the parallel component of the particle velocity $\mathbf{v}$, and the angle brackets denoting integration over gyro-angle ϑ and the perpendicular component of the particle velocity $v_\perp$.

As described in, e.g., Report 2047537-TN-05-1 [2], the drift kinetic equations describing the evolution of the marginalised particle distribution f_s for the ions and neutrals are

$$\frac{\partial f_i}{\partial t} + v_\parallel \frac{\partial f_i}{\partial z} - \frac{e}{m_i}\frac{\partial \phi}{\partial z}\frac{\partial f_i}{\partial v_\parallel} = -R_{\text{in}}\left(n_n f_i - n_i f_n\right) + R_{\text{ion}} n_e f_n + S_i \tag{3}$$

and

$$\frac{\partial f_n}{\partial t} + v_\parallel \frac{\partial f_n}{\partial z} = R_{\text{in}}\left(n_n f_i - n_i f_n\right) - R_{\text{ion}} n_e f_n + S_n, \tag{4}$$

where ϕ is the electrostatic potential, $n_s = \int dv_\parallel f_s$ is the particle density, $S_s = \left\langle \hat{S}_s \right\rangle$ is a marginalised source accounting for, e.g., heating and fueling, and R_{in} and R_{ion} are constants that determine the ion-neutral charge exchange and ionisation rates,

respectively. The distribution function f_i is related to the electron density n_e via quasineutrality:

$$n_i = \int_{-\infty}^{\infty} dv_\parallel f_i = n_e. \tag{5}$$

One of the main challenges in solving this system of equations is that there is no explicit equation for the electrostatic potential. If one were to try, e.g., to solve the ion drift kinetic equation (3) and a corresponding electron drift kinetic equation using an explicit time advance algorithm with ϕ at the previous time level as an input, then in general the solutions for f_i and f_e so obtained would not satisfy Eq. (5). This procedure could be iterated, with ϕ varied until quasineutrality were satisfied, or one could develop an approach that guarantees satisfaction of quasineutrality from the outset. We have presented two such approaches in previous reports: The first assumes a Boltzmann electron response and thus avoids entirely the need to solve for the electron dynamics, while the second uses a novel moment-kinetic approach in which the electron parallel momentum equation can be used to solve for the potential explicitly while enforcing quasineutrality. In this report we describe the numerical implementation and testing of a set of electron fluid equations that can be used as the basis for the moment-kinetic approach alluded to above.

We will assume in this report that f_i and f_n can be obtained, provided ϕ, either via direct solution of the above drift kinetic equations or via the moment-kinetic approach derived in report 2047357-TN-05-01 [2]. Our focus here will be on describing the numerical treatment of the electron fluid equations and their relationship to the calculation of the electrostatic potential.

3. Electron fluid equations

Electron fluid equations are obtained by taking the appropriate velocity moments of the electron drift kinetic equation. Details can be found in [2]. The electron continuity equation is‡

$$\frac{\partial n_e}{\partial t} + \frac{\partial n_e u_e}{\partial z} = n_e n_n R_{\text{ion}} + \int_{-\infty}^{\infty} dv_\parallel S_e, \tag{6}$$

where the parallel flow of species s is given by

$$u_s \doteq \frac{1}{n_s} \int_{-\infty}^{\infty} dv_\parallel v_\parallel f_s. \tag{7}$$

The charge conservation equation, obtained by combining the electron and ion continuity equations and enforcing quasineutrality, is

$$\frac{\partial}{\partial z} \left(n_e \left(u_i - u_e \right) \right) = 0. \tag{8}$$

‡ Here we have corrected a typo in the sign of the electron ionisation particle source appearing in [2].

Note that we have assumed $\int d^3v\,(S_i - S_e)$ is small to obtain the above result. The electron parallel momentum equation is

$$-\frac{\partial p_{\parallel,e}}{\partial z} + en_e\frac{\partial \phi}{\partial z} + F_\parallel[f_e, f_i] + n_e m_e n_n R_{\rm en}\,(u_n - u_e) = 0, \tag{9}$$

where $p_{\parallel,e}$ is the electron parallel pressure,

$$p_{\parallel,e} \doteq \int d^3v\; m_e w_\parallel^2 f_e, \tag{10}$$

$w_\parallel \doteq v_\parallel - u_e$ is the parallel component of the electron peculiar velocity, $R_{\rm en}$ is a constant that determines the electron-neutral elastic collision rate, and

$$F_\parallel = F_\parallel[f_e, f_i](z,t) \doteq \int d^3v\; m_e v_\parallel C_{ei}[\hat{f}_e, \hat{f}_i] \tag{11}$$

is the parallel friction force between electrons and ions, with C_{ei} the electron-ion collision operator. The term proportional to S_e has been neglected in (9) as small in m_e/m_i because we order $S_e \sim f_e v_{\rm th,i}/L_z$, with $v_{\rm th,i} \doteq \sqrt{2T_i/m_i}$.

Finally, the electron energy equation is

$$\begin{aligned}\frac{3}{2}n_e\left(\frac{\partial T_e}{\partial t} + u_e\frac{\partial T_e}{\partial z}\right) =& -\left(\frac{\partial q_{\parallel,e}}{\partial z} + p_{\parallel,e}\frac{\partial u_e}{\partial z}\right) + \frac{3n_e m_e \nu_{ei}}{m_i}\,(T_i - T_e) + F_\parallel\,(u_i - u_e)\\ &+ \frac{3n_e m_e n_n R_{\rm en}}{m_i}\,(T_n - T_e) + n_e m_e n_n R_{\rm en}\,(u_n - u_e)^2\\ &+ \int d^3v\left(\frac{m_e w^2}{2} - \frac{3}{2}T_e\right)\hat{S}_e - n_e n_n R_{\rm ion} E_{\rm ion},\end{aligned} \tag{12}$$

where $w \doteq |\mathbf{v} - u_e\hat{\mathbf{z}}|$ is the peculiar speed, $E_{\rm ion}$ is the ionisation energy cost (including radiation from excited states), and $q_{\parallel,e}$ is the electron parallel heat flux,

$$q_{\parallel,e} \doteq \int d^3w m_s w_\parallel^3 f_e. \tag{13}$$

The electron energy equation (12) can also be expressed in terms of the electron pressure:

$$\begin{aligned}\frac{\partial p_e}{\partial t} =& -\frac{2}{3}\frac{\partial q_{\parallel,e}}{\partial z} - \left(\frac{2}{3}p_{\parallel,e} + p_e\right)\frac{\partial u_e}{\partial z} - u_e\frac{\partial p_e}{\partial z} + \frac{2n_e m_e \nu_{ei}}{m_i}\,(T_e - T_i)\\ &+ \frac{2}{3}F_\parallel\,(u_i - u_e) + \frac{2n_e m_e n_n R_{\rm en}}{m_i}\,(T_n - T_e) + \frac{2}{3}n_e m_e n_n R_{\rm en}\,(u_n - u_e)^2\\ &+ n_n R_{\rm ion}\left(p_e - \frac{2}{3}n_e E_{\rm ion}\right) + \int d^3v\frac{m_e w^2}{3}\hat{S}_e.\end{aligned} \tag{14}$$

To facilitate our 1+1D treatment, we assume that the particle distribution $\hat{f}_s$ and source $\hat{S}_s$ are isotropic, so that $p_s = p_{\parallel,s}$ and $\int d^3v(m_s w^2/3)\hat{S}_s = \int dv_\parallel m_s w_\parallel^2 S_s$. With

these approximations, the electron energy equation is

$$\begin{aligned}\frac{\partial p_{\parallel,e}}{\partial t} =& -\frac{2}{3}\frac{\partial q_{\parallel,e}}{\partial z} - \frac{5}{3}p_{\parallel,e}\frac{\partial u_e}{\partial z} - u_e\frac{\partial p_{\parallel,e}}{\partial z} + \frac{2m_e\nu_{ei}}{m_i n_i}\left(n_i p_{\parallel,e} - n_e p_{\parallel,i}\right) \\ &+ \frac{2}{3}F_\parallel\left(u_i - u_e\right) + \frac{2m_e R_{\mathrm{en}}}{m_i}\left(n_e p_{\parallel,n} - n_n p_{\parallel,e}\right) + \frac{2}{3}n_e m_e n_n R_{\mathrm{en}}\left(u_n - u_e\right)^2 \\ &+ n_n R_{\mathrm{ion}}\left(p_{\parallel,e} - \frac{2}{3}n_e E_{\mathrm{ion}}\right) + \int_{-\infty}^{\infty} dv_\parallel m_e w_\parallel^2 S_e.\end{aligned} \tag{15}$$

The set of fluid equations (8)-(11) and (13)-(15) require some form of closure to calculate the parallel friction and heat flux. Ultimately, we intend to use the moment-kinetic treatment proposed in [2], but for this report we employ the simple Braginskii closure [3]:

$$F_\parallel = 0.51 n_e m_e \nu_{ei}\left(u_i - u_e\right) - 0.71 n_e\frac{\partial T_e}{\partial z} \tag{16}$$

and

$$q_{\parallel,e} = -3.16\frac{p_{\parallel,e}}{m_e\nu_{ei}}\frac{\partial T_e}{\partial z} - 0.71 p_{\parallel,e}\left(u_i - u_e\right). \tag{17}$$

The set of equations (3)-(5), (8), (9), (15) and (16)-(17) constitute a closed set of equations for f_i, f_n, ϕ, n_e, u_e and $p_{\parallel,e}$, provided an appropriate set of boundary conditions. We turn our attention to these boundary conditions next.

4. Boundary conditions

To ensure uniqueness of our solution, we must specify boundary conditions in z on the ion and neutral particle distributions, and on the electron parallel flow and pressure. For the ions, we assume that all ions that reach the end of the domain escape to the wall, where they recombine. Thus, no ions return; i.e.,

$$f_i(z_+, v_\parallel < 0, t) = 0, \tag{18}$$

and

$$f_i(z_-, v_\parallel > 0, t) = 0. \tag{19}$$

Neutrals that leave the domain are assumed to hit the wall and thermalise at the temperature of the wall, T_w. Ions that recombine at the wall also re-enter as neutrals. The resulting boundary condition on the neutrals is

$$f_n(z_-, v_\parallel > 0, t) = \Gamma_- f_{Kw}(v_\parallel), \tag{20}$$

and

$$f_n(z_+, v_\parallel < 0, t) = \Gamma_+ f_{Kw}(v_\parallel), \tag{21}$$

where

$$f_{Kw}(v_\parallel) \doteq 3\sqrt{\pi}\left(\frac{m_i}{2T_w}\right)^{3/2}\left|v_\parallel\right|\mathrm{erfc}\left(\sqrt{\frac{m_i}{2T_w}}\left|v_\parallel\right|\right) \tag{22}$$

is the (marginalised) Knudsen cosine distribution [4], and

$$\Gamma_- \doteq \sum_{s=i,n} \int_{-\infty}^{0} dv_\parallel \left|v_\parallel\right| f_s(z_-, v_\parallel, t) \tag{23}$$

and

$$\Gamma_+ \doteq \sum_{s=i,n} \int_{0}^{\infty} dv_\parallel \left|v_\parallel\right| f_s(z_+, v_\parallel, t) \tag{24}$$

are the combined fluxes of neutrals and ions towards the walls at $z = z_-$ and $z = z_+$, respectively.

For electrons that leave the domain, their parallel energy $\mathcal{E}_\parallel = m_e v_\parallel^2/2 - e\phi$ is conserved. As a result, electrons with parallel speeds greater than $v_{c,+} \doteq \sqrt{2e(\phi(z_+,t) - \phi_w)/m_e}$ at z_+ reach the wall, with ϕ_w the potential of the wall beyond $z = z_+$; electrons with speeds less than $v_{c,+}$ are repelled back into the domain:

$$f_e(z_+, v_\parallel < 0, t) = \begin{cases} f_e(z_+, -v_\parallel, t), & |v_\parallel| < v_{c,+} \\ 0, & |v_\parallel| > v_{c,+}. \end{cases} \tag{25}$$

The boundary condition at $z = z_-$ is similar:

$$f_e(z_-, v_\parallel > 0, t) = \begin{cases} f_e(z_-, -v_\parallel, t), & |v_\parallel| < v_{c,-} \\ 0, & |v_\parallel| > v_{c,-}, \end{cases} \tag{26}$$

where $v_{c,-} \doteq \sqrt{2e\phi(z_-,t)/m_e}$, and we have chosen ϕ to be zero at the wall beyond z_-.

Integrating the charge conservation equation (8) and using (25)-(26) gives a constraint on the parallel current at the domain boundaries:

$$\begin{aligned} 0 = J_\parallel(z_+,t) - J_\parallel(z_-,t) = &\int_0^\infty dv_\parallel v_\parallel f_i(z_+, v_\parallel, t) - \int_{v_{c,+}}^\infty dv_\parallel v_\parallel f_e(z_+, v_\parallel, t) \\ &- \int_{-\infty}^0 dv_\parallel v_\parallel f_i(z_-, v_\parallel, t) + \int_{-\infty}^{-v_{c,-}} dv_\parallel v_\parallel f_e(z_-, v_\parallel, t). \end{aligned} \tag{27}$$

Note that there is no contribution to the electron current from particles with $|v_\parallel| < v_{c,+}$ at $z = z_+$ nor from particles with $|v_\parallel| < v_{c,-}$ at $z = z_-$. This is because the outgoing current of electrons with these speeds is cancelled by the return current of these electrons.

As proposed in [2], we simplify our system by assuming that the parallel current into the wall vanishes individually at each boundary; i.e., $J_\parallel(z_+,t) = J_\parallel(z_-,t) = 0$. Combined with quasineutrality, this imposes $u_e(z_\pm,t) = u_i(z_\pm,t)$. The charge conservation equation (8) can then be integrated to find $u_e = u_i$ for all z.

Our boundary condition on $p_{\parallel,e}$ is chosen based on the physics we intend to model; for the cases considered in this report, we either set $T_e(z_\pm,t) = T_i(z_\pm,t)$ (when ν_{ei} is large) or we assume that the electron temperature is constant throughout the sheath so that $T_e(z_\pm,t) = T_w$ (when we wish to force a Boltzmann electron response), with T_w the temperature of the wall.

5. Normalisations

The normalisations used in the code are given in Table 1.

The normalised drift kinetic equations for ions and neutrals are

$$\frac{\partial \tilde{f}_i}{\partial \tilde{t}} + \tilde{v}_\parallel \frac{\partial \tilde{f}_i}{\partial \tilde{z}} - \frac{1}{2}\frac{\partial \tilde{\phi}}{\partial \tilde{z}}\frac{\partial \tilde{f}_i}{\partial \tilde{v}_\parallel} = -\tilde{R}_{\mathrm{in}}\left(\tilde{n}_n \tilde{f}_i - \tilde{n}_i \tilde{f}_n\right) + \tilde{R}_{\mathrm{ion}}\tilde{n}_e \tilde{f}_n + \tilde{S}_i \tag{28}$$

and

$$\frac{\partial \tilde{f}_n}{\partial \tilde{t}} + \tilde{v}_\parallel \frac{\partial \tilde{f}_n}{\partial \tilde{z}} = \tilde{R}_{\mathrm{in}}\left(\tilde{n}_n \tilde{f}_i - \tilde{n}_i \tilde{f}_n\right) - \tilde{R}_{\mathrm{ion}}\tilde{n}_e \tilde{f}_n + \tilde{S}_n. \tag{29}$$

The normalised electron fluid equations are

$$\tilde{n}_e = \tilde{n}_i, \tag{30}$$

$$\frac{\partial}{\partial \tilde{z}}\left(\tilde{n}_e\left(\tilde{u}_i - \tilde{u}_e\right)\right) = 0, \tag{31}$$

$$-\frac{\partial \tilde{p}_{\parallel,e}}{\partial \tilde{z}} + \frac{\tilde{n}_e}{2}\frac{\partial \tilde{\phi}}{\partial \tilde{z}} + \tilde{F}_\parallel + \tilde{n}_e \tilde{m}_e \tilde{n}_n \tilde{R}_{\mathrm{en}}\left(\tilde{u}_n - \tilde{u}_e\right) = 0, \tag{32}$$

and

$$\begin{aligned}\frac{\partial \tilde{p}_{\parallel,e}}{\partial \tilde{t}} =& -\frac{2}{3}\frac{\partial \tilde{q}_{\parallel,e}}{\partial \tilde{z}} - \frac{5}{3}\tilde{p}_{\parallel,e}\frac{\partial \tilde{u}_e}{\partial z} - \tilde{u}_e\frac{\partial \tilde{p}_{\parallel,e}}{\partial \tilde{z}} + 2\tilde{m}_e\tilde{\nu}_{ei}\left(\tilde{p}_{\parallel,i} - \tilde{p}_{\parallel,e}\right) \\ &+ \frac{2}{3}\tilde{F}_\parallel\left(\tilde{u}_i - \tilde{u}_e\right) + 2\tilde{m}_e\tilde{R}_{\mathrm{en}}\left(\tilde{n}_e\tilde{p}_{\parallel,n} - \tilde{n}_n\tilde{p}_{\parallel,e}\right) + \frac{2}{3}\tilde{n}_e\tilde{m}_e\tilde{n}_n\tilde{R}_{\mathrm{en}}\left(\tilde{u}_n - \tilde{u}_e\right)^2 \\ &+ \tilde{n}_n\tilde{R}_{\mathrm{ion}}\left(\tilde{p}_{\parallel,e} - \frac{2}{3}\tilde{n}_e\tilde{E}_{\mathrm{ion}}\right) + \tilde{\mathcal{P}}_e,\end{aligned} \tag{33}$$

where

$$\tilde{\mathcal{P}}_e \doteq \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \tilde{v}_\parallel \tilde{m}_e \tilde{w}_\parallel^2 \tilde{S}_e. \tag{34}$$

The zero current boundary condition at the wall, combined with quasineutrality (30) and charge conservation (31), imposes $\tilde{u}_e = \tilde{u}_i$.

6. Numerical approach

A detailed description of the space and time discretisations employed in the code can be found in report 2047357-TN-14 [5]. Briefly, we employ an explicit time advance algorithm (a strong-stability-preserving Runge-Kutta variant) to evolve the ion and neutral particle distribution functions, as well as the electron parallel pressure. A Chebyshev spectral element scheme is used for the spatial discretisation. Solution of the coupled equations proceeds schematically in the following way:

- Solve the ion (28) and neutral (29) drift kinetic equations for f_i and f_n within a Runge-Kutta stage, given the parallel electric field $E_\parallel$ at the previous time step, and subject to the appropriate boundary conditions (18)-(21).

normalised variable	definition
$\tilde{t}$	$t(c_{\rm ref}/L_{\rm ref})$
$\tilde{z}$	$z/L_{\rm ref}$
$\tilde{v}_\parallel$	$v_\parallel/c_s$
$\tilde{w}$	w/c_s
$\tilde{n}_s$	$n_s/n_{\rm ref}$
$\tilde{u}_s$	$u_s/c_{\rm ref}$
$\tilde{p}_{\parallel,s}$	$p_{\parallel,s}/(2n_{\rm ref}T_{\rm ref})$
$\tilde{q}_{\parallel,e}$	$q_{\parallel,s}/(2n_{\rm ref}T_{\rm ref}c_{\rm ref})$
$\tilde{T}_s$	$T_s/T_{\rm ref}$
$\tilde{\phi}$	$e\phi/T_{\rm ref}$
$\tilde{E}_z$	$eL_{\rm ref}E_z/T_{\rm ref}$
$\tilde{R}_{\rm sn}$	$R_{\rm sn}(n_{\rm ref}L_{\rm ref}/c_{\rm ref})$
$\tilde{R}_{\rm ion}$	$R_{\rm ion}(n_{\rm ref}L_{\rm ref}/c_{\rm ref})$
$\tilde{E}_{\rm ion}$	$E_{\rm ion}/(2T_{\rm ref})$
$\tilde{\nu}_{ei}$	$\nu_{ei}(L_{\rm ref}/c_{\rm ref})$
$\tilde{f}_s$	$f_s(c_s\pi^{1/2}/n_{\rm ref})$
$\tilde{F}_\parallel$	$F_\parallel(L_{\rm ref}/2n_{\rm ref}T_{\rm ref})$
$\tilde{S}_s$	$S_s(L_{\rm ref}\pi^{1/2}/n_{\rm ref})(c_s/c_{\rm ref})$
$\tilde{m}_e$	m_e/m_i

reference quantity	definition
$L_{\rm ref}=L_z$	ref. length
$T_{\rm ref}$	ref. temperature
$n_{\rm ref}$	ref. density
$c_{\rm ref}$	$\sqrt{2T_{\rm ref}/m_i}$
m_i	ion mass

Table 1. Definitions for normalised and reference quantities used in the report. Note that $c_s = c_{\rm ref}\sqrt{m_i/m_s}$.

- Solve the electron energy equation (33) to update $p_{\parallel,e}$ within a Runge-Kutta stage.
- With f_i and f_n updated, calculate the ion density and parallel flow, and set $n_e = n_i$ and $u_e = u_i$ at the new Runge-Kutta stage.
- Solve the electron parallel momentum equation (32) for the parallel electric field $E_\parallel$.
- Repeat.

7. Numerical results

We first provide numerical results testing the code implementation and then provide preliminary results for a more general case.

7.1. Boltzmann electron test

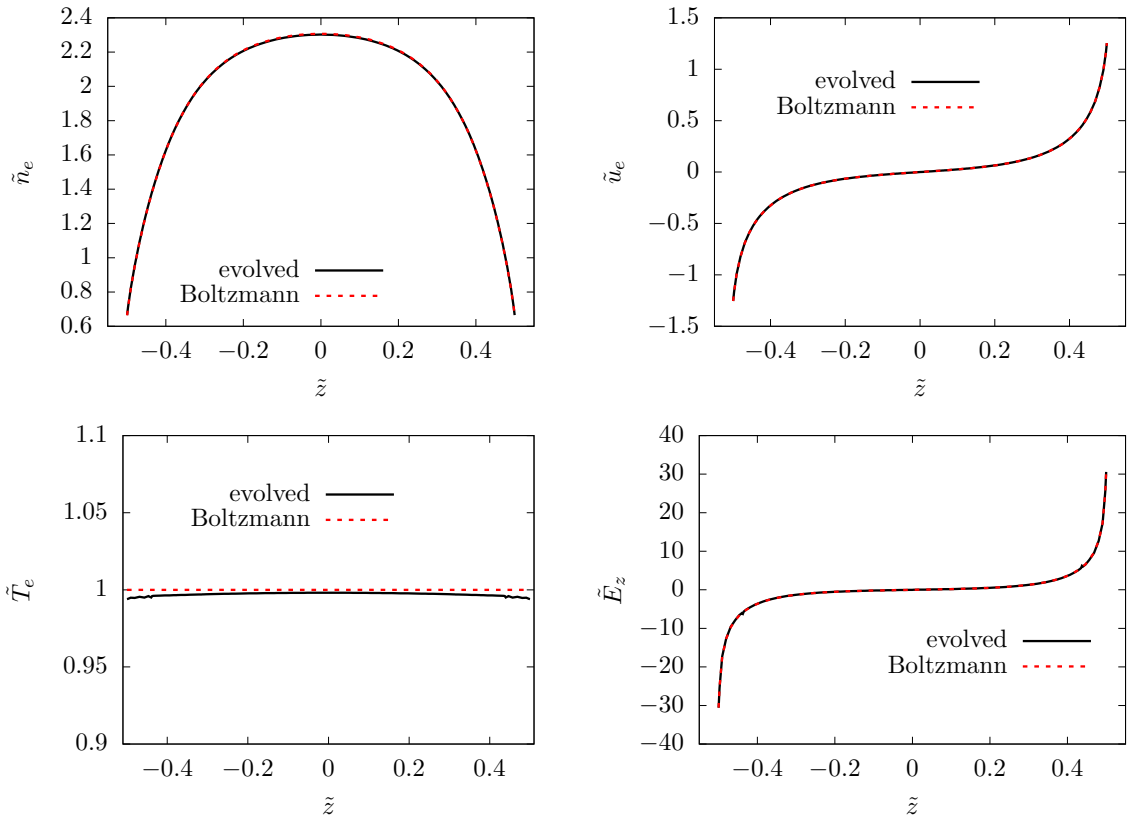

Figure 1. Comparison of simulation data for a pair of simulations, one of which has a hard-wired Boltzmann response for the electrons ("Boltzmann") and the other ("evolved") solves the electron fluid equations outlined in this report. Clockwise from top left to bottom left is the electron density, electron parallel flow, parallel electric field and electron temperature.

For the Braginskii fluid closure we are employing, whenever $\partial\tilde{T}_e/\partial\tilde{z} = 0 = \tilde{R}_{\rm en}$, the electrostatic potential that satisfies the electron parallel momentum equation (32) is $\tilde{\phi} = \tilde{T}_e \ln \tilde{n}_e$; i.e., there is a Boltzmann electron density response. This is because the parallel friction $\tilde{F}_\parallel$ vanishes (recall that $\tilde{u}_e = \tilde{u}_i$ everywhere due to the zero parallel current boundary condition). A homogeneous electron temperature can be assured at all times by judicious choice of $\mathcal{P}_e$ and a homogeneous initial condition for $\tilde{T}_e$. The necessary $\mathcal{P}_e$ (with $\tilde{R}_{\rm en} = 0$) is

$$\tilde{\mathcal{P}}_e = \frac{2}{3}\tilde{p}_{\parallel,e}\frac{\partial\tilde{u}_e}{\partial z} - 2\tilde{m}_e\tilde{\nu}_{ei}\left(\tilde{p}_i - \tilde{p}_e\right) + \frac{2}{3}\tilde{n}_n\tilde{R}_{\rm ion}\tilde{n}_e\tilde{E}_{\rm ion}, \tag{35}$$

Using this source, initialising with a homogeneous electron temperature and neglecting electron charge exchange collisions, we obtain the results given in Fig. 1. Input files with the numerical parameters used can be found in [6] and [7]. The excellent agreement with a simulation run with a hard-wired Boltzmann electron response gives us confidence that our numerical implementation is working properly.

7.2. equilibrium with Braginskii closure

Sample results from a simulation with large electron-ion collision frequency – for which we expect $T_e = T_i$ – are given in Fig. 2. We see that indeed $T_e = T_i$ to a good approximation across most of the z domain. The results actually look quite similar to those from simulations with an assumed Boltzmann electron response. This is not entirely surprising, as the ion temperature profile is largely flat and $u_i = u_e$ so that the only deviation from a Boltzmann response present in the electron parallel momentum equation is the electron-neutral charge exchange. There is some roughness in the parallel electric field profile (see the bottom right plot in Fig. 2) that could use further exploration, but otherwise the profiles are fairly smooth and the simulation well-behaved. Inputs for this simulation can be found at [8].

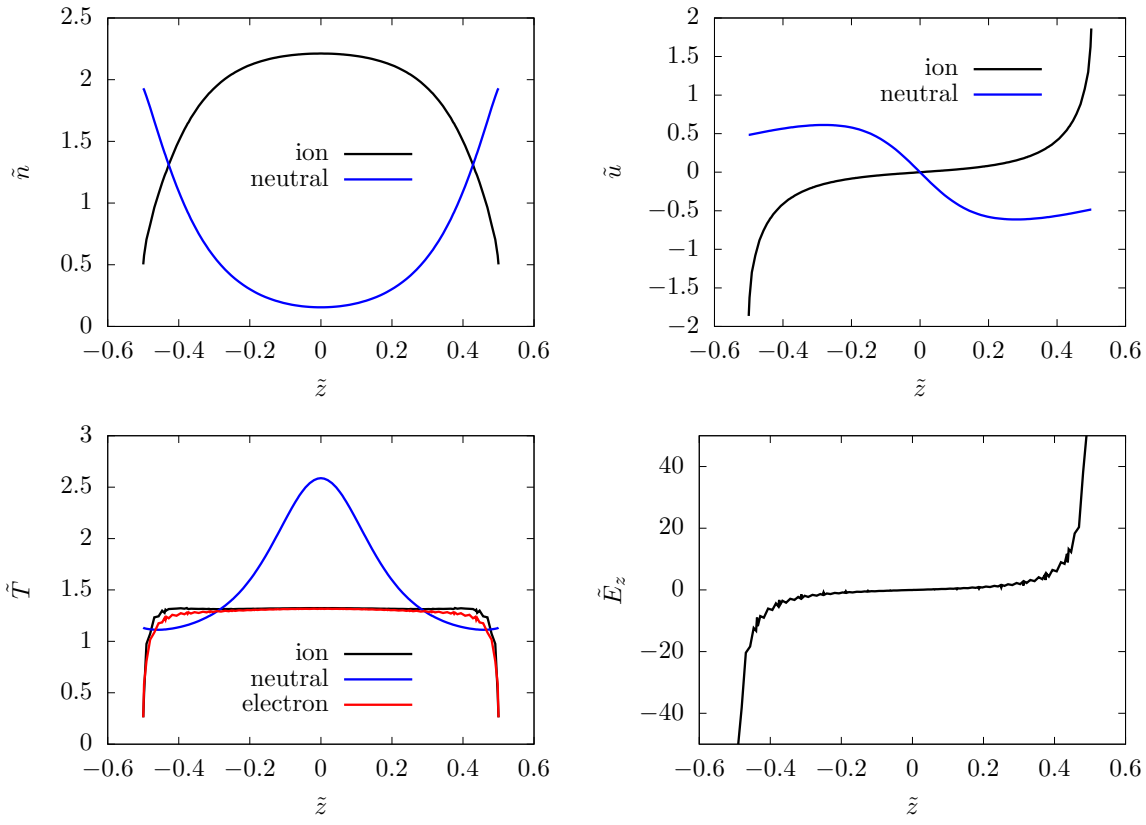

Figure 2. Steady-state moments of the ion, neutral and electron distribution functions for a simulation with all collisional terms included (and notably, $\tilde{\nu}_{ei} = 10^5$) and the Braginskii fluid closure.

8. Discussion

With this report we have demonstrated that our code now has a functional, time-dependent fluid model for electron dynamics that can be used to calculate the

electrostatic potential. Our immediate aim going forward is to improve the closure for this fluid model; in particular, we hope to use the moment-kinetic approach proposed in [2] in place of the Braginskii closure employed here. This will be significant in that it will introduce the need to solve for both the electron particle distribution function and the electrostatic potential itself (rather than the parallel electric field that we solved for in this report), a non-trivial complication both conceptually and numerically.

Appendix A. Supporting documentation for the simulations

The simulations used to create the data presented in this report were generated by the branch `https://github.com/mabarnes/moment_kinetics/tree/electrons`, with the latest commit at the time of writing being `9b68be0`.

In this appendix we give URL links to the input files used to generate the simulation data. To run a simulation use the following command:

```
$ julia -O3 --project run_moment_kinetics.jl input.toml
```

with `input.toml` replaced by the appropriate input file name.

The input files for the Boltzmann response test can be found at [7] (for the run with the Braginskii closure and heat source needed to force a Boltzmann solution to the electron fluid equations) and [6] (for the run with a built-in Boltzmann electron response). For the simulation with Braginskii closure and large ν_{ei} (corresponding to Fig. 2), the input file can be found at [8].

[1] F. I. Parra, M. Barnes, and M. R. Hardman. 2d drift kinetic model with wall boundary conditions. *Excalibur/Neptune Report*, 7:2047357–TN–07–02 M1.4, 2021.

[2] F. I. Parra, M. Barnes, and M. R. Hardman. 1d drift kinetic models with wall boundary conditions. *Excalibur/Neptune Report*, 5:2047357–TN–05–01 M1.3, 2021.

[3] S. I. Braginskii. Transport phenomena in a completely ionized two-temperature plasma. *Sov. Phys. JETP*, 6:358, 1958.

[4] M. Knudsen. Das cosinusgesetz in der kinetischen gastheorie. *Annal. Phys.*, 353:1113, 1916.

[5] M. Barnes, F. I. Parra, M. R. Hardman, and J. Omotani. Numerical study of 1+1d drift kinetic models for parallel dynamics in the plasma edge. *Excalibur/Neptune Report*, 14:2047357–TN–14 D2.2, 2022.

[6] `https://github.com/mabarnes/moment_kinetics/blob/electrons/runs/wall%2Bsheath-bc_boltzmann.toml`.

[7] `https://github.com/mabarnes/moment_kinetics/blob/electrons/runs/wall%2Bsheath-bc_braginskii_boltzmann_test.toml`.

[8] `https://github.com/mabarnes/moment_kinetics/blob/electrons/runs/wall%2Bsheath-bc_braginskii_colls.toml`.