Numerical study of 1+1D, moment-based drift kinetic models with periodic boundary conditions

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1. Introduction

We expect that one of the biggest challenges in numerically solving drift kinetic equations in the plasma edge is treating the motion of electrons along the magnetic field. Because the electrons are light, they move rapidly along the field, placing a severe stability restriction on the step size for explicit time advance schemes. Unfortunately, an implicit treatment is not straightforward due to an implicit dependence of the electrostatic potential on the charged particle distribution functions. One of the main aims of our research is to develop and test a novel analytical model and associated numerical algorithm for relaxing this restriction. As a first step towards this goal, we developed a new code in the programming language Julia to simulate a simple drift kinetic model for parallel dynamics [1]. We then extended the code to simulate a modified set of equations in which the density is removed from the particle distribution function and is evolved separately using the continuity equation [2]. In this report we describe the numerical implementation of the full 'moment-kinetic' model, in which the particle density, parallel flow and parallel pressure are evolved separately from a modified particle distribution function. Numerical results are presented to demonstrate that the relevant conservation properties are satisfied to machine precision and that the moment-based approach passes the linear damping benchmark developed in [1, 3].

2. Model equations

A detailed derivation of the drift kinetic model we consider, as well as the full moment-based model, is provided in our Jan 2021 report [3]. For the Reader's convenience we produce first an overview of the drift kinetic model and then of models in which combinations of the density, parallel flow and parallel pressure are separately evolved. The system we consider consists of a single ion species of charge e, a single neutral species, and an electron species modelled as having a Boltzmann response, all immersed in a straight, uniform magnetic field in the z direction. We allow for charge exchange collisions between ions and neutrals but do not account for intra-species collisions. Finally, we assume that the plasma is homogeneous in the plane perpendicular to the magnetic field. With these assumptions, our model system of equations is

$$\frac{\partial f_i}{\partial t} + v_\parallel \frac{\partial f_i}{\partial z} - \frac{e}{m_i} \frac{\partial \phi}{\partial z} \frac{\partial f_i}{\partial v_\parallel} = -R_{in} \left(n_n f_i - n_i f_n \right), \tag{1}$$

$$\frac{\partial f_n}{\partial t} + v_\parallel \frac{\partial f_n}{\partial z} = -R_{in} \left(n_i f_n - n_n f_i \right), \tag{2}$$

$$n_s(z,t) = \int_{-\infty}^{\infty} dv_\parallel f_s(z, v_\parallel, t), \tag{3}$$

and

$$n_i = N_e \exp\left(\frac{e\phi}{T_e} \right), \tag{4}$$

with $f_s \doteq \int d\vartheta dv_\perp v_\perp F_s$ the marginalized particle distribution function for species s, $v_\parallel$ and $v_\perp$ the components of the particle velocity parallel and perpendicular to the magnetic field, respectively, ϑ the gyro-angle, m_i the ion mass, t the time, ϕ the electrostatic potential, and R_{in} a charge exchange collision frequency factor.

For our boundary conditions, we impose periodicity on f_s in both z and $v_\parallel$, with periods L_z and $L_{v_\parallel}$, respectively. There is also the option to impose zero boundary conditions on z and $v_\parallel$ at the upwind boundary of the domain. As f_s should go to zero at $v_\parallel \to \pm\infty$, imposition of zero boundary conditions and periodic boundary conditions should be equivalent as long as $L_{v_\parallel}$ is sufficiently large. Note that with either choice of boundary conditions, the line-averaged density $\int_0^{L_z} dz\ n_s$ should be conserved.

We normalize Eqs. (1)-(4) by defining

$$\tilde{f}_s \doteq f_s \frac{c_s \sqrt{\pi}}{N_e}, \tag{5}$$

$$\tilde{t} \doteq t \frac{c_s}{L_z}, \tag{6}$$

$$\tilde{z} \doteq \frac{z}{L_z}, \tag{7}$$

$$\tilde{v}_\parallel \doteq \frac{v_\parallel}{c_s}, \tag{8}$$

$$\tilde{n}_s \doteq \frac{n_s}{N_e}, \tag{9}$$

$$\tilde{\phi} \doteq \frac{e\phi}{T_e}, \tag{10}$$

and

$$\tilde{R}_{\text{in}} \doteq R_{\text{in}} \frac{N_e L_z}{c_s} \tag{11}$$

with $c_s \doteq \sqrt{2T_e/m_s}$. In terms of these normalised quantities, Eqs (1)-(4) become

$$\frac{\partial \tilde{f}_i}{\partial \tilde{t}} + \tilde{v}_\parallel \frac{\partial \tilde{f}_i}{\partial \tilde{z}} - \frac{1}{2}\frac{\partial \tilde{\phi}}{\partial \tilde{z}}\frac{\partial \tilde{f}_i}{\partial \tilde{v}_\parallel} = -\tilde{R}_{\rm in}\left(\tilde{n}_n \tilde{f}_i - \tilde{n}_i \tilde{f}_n\right), \tag{12}$$

$$\frac{\partial \tilde{f}_n}{\partial \tilde{t}} + \tilde{v}_\parallel \frac{\partial \tilde{f}_n}{\partial \tilde{z}} = -\tilde{R}_{\rm in}\left(\tilde{n}_i \tilde{f}_n - \tilde{n}_n \tilde{f}_i\right), \tag{13}$$

$$e^{\tilde{\phi}} = \tilde{n}_i = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\tilde{v}_\parallel \tilde{f}_i, \tag{14}$$

and

$$\tilde{n}_n = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\tilde{v}_\parallel \tilde{f}_n. \tag{15}$$

2.1. Moment approach: density

We now define the modified distribution function $g_s \doteq f_s/n_s$ so that $\int dv_\parallel g_s = 1$. In terms of g_s, the system of equations given by Eqs. (1)-(4) becomes

$$n_i\left(\frac{\partial g_i}{\partial t} + v_\parallel \frac{\partial g_i}{\partial z} - \frac{e}{m_i}\frac{\partial \phi}{\partial z}\frac{\partial g_i}{\partial v_\parallel}\right) + g_i\left(\frac{\partial n_i}{\partial t} + v_\parallel \frac{\partial n_i}{\partial z}\right) = -R_{\rm in} n_i n_n \left(g_i - g_n\right), \tag{16}$$

$$n_n\left(\frac{\partial g_n}{\partial t} + v_\parallel \frac{\partial g_n}{\partial z}\right) + g_n\left(\frac{\partial n_n}{\partial t} + v_\parallel \frac{\partial n_n}{\partial z}\right) = -R_{\rm in} n_i n_n \left(g_n - g_i\right), \tag{17}$$

$$n_i = N_e \exp\left(\frac{e\phi}{T_e}\right), \tag{18}$$

$$\frac{\partial n_s}{\partial t} + \frac{\partial n_s u_s}{\partial z} = 0, \tag{19}$$

and

$$u_s = \int_{-\infty}^{\infty} dv_\parallel g_s v_\parallel. \tag{20}$$

Note that the 1D continuity equation (19) has replaced the moment equation (3) as a means of computing the density for each species.

Substituting the continuity equation (19) into the drift kinetic equations (16) and (17) gives

$$\frac{\partial g_i}{\partial t} + v_\parallel \frac{\partial g_i}{\partial z} - \frac{e}{m_i}\frac{\partial \phi}{\partial z}\frac{\partial g_i}{\partial v_\parallel} = -R_{\rm in} n_n \left(g_i - g_n\right) + g_i\left(\frac{\partial u_i}{\partial z} - \left(v_\parallel - u_i\right)\frac{\partial \ln n_i}{\partial z}\right) \tag{21}$$

and

$$\frac{\partial g_n}{\partial t} + v_\parallel \frac{\partial g_n}{\partial z} = -R_{\rm in} n_i \left(g_n - g_i\right) + g_n\left(\frac{\partial u_n}{\partial z} - \left(v_\parallel - u_n\right)\frac{\partial \ln n_n}{\partial z}\right), \tag{22}$$

We normalize Eqs. (18)-(22) by using Eqs. (6)-(11) and by further defining

$$\tilde{g}_s \doteq g_s c_s \sqrt{\pi} \tag{23}$$

and

$$\tilde{u}_s \doteq \frac{u_s}{c_s}. \tag{24}$$

In terms of these normalised quantities, Eqs (18)-(22) become

$$\frac{\partial \tilde{g}_i}{\partial \tilde{t}} + \tilde{v}_\parallel \frac{\partial \tilde{g}_i}{\partial \tilde{z}} - \frac{1}{2}\frac{\partial \tilde{\phi}}{\partial \tilde{z}}\frac{\partial \tilde{g}_i}{\partial \tilde{v}_\parallel} = -\tilde{R}_{\rm in}\tilde{n}_n\left(\tilde{g}_i - \tilde{g}_n\right) + \tilde{g}_i\left(\frac{\partial \tilde{u}_i}{\partial \tilde{z}} - \left(\tilde{v}_\parallel - \tilde{u}_i\right)\frac{\partial \ln \tilde{n}_i}{\partial \tilde{z}}\right), \tag{25}$$

$$\frac{\partial \tilde{g}_n}{\partial \tilde{t}} + \tilde{v}_\parallel \frac{\partial \tilde{g}_n}{\partial \tilde{z}} = -\tilde{R}_{\rm in}\tilde{n}_i\left(\tilde{g}_n - \tilde{g}_i\right) + \tilde{g}_n\left(\frac{\partial \tilde{u}_n}{\partial \tilde{z}} - \left(\tilde{v}_\parallel - \tilde{u}_n\right)\frac{\partial \ln \tilde{n}_n}{\partial \tilde{z}}\right), \tag{26}$$

$$\frac{\partial \tilde{n}_s}{\partial \tilde{t}} + \frac{\partial \tilde{n}_s \tilde{u}_s}{\partial \tilde{z}} = 0, \tag{27}$$

$$e^{\tilde{\phi}} = \tilde{n}_i, \tag{28}$$

and

$$\tilde{u}_s = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\tilde{v}_\parallel \tilde{g}_n \tilde{v}_\parallel. \tag{29}$$

The above form for the equations is appealing because it maintains the form of an advection equation with the only modification being the addition of source terms. However, it can pose challenges for numerical conservation of quantities such as the 0th velocity moment of g_s. This is because parts of the source terms must cancel upon velocity space integration with some of the advective terms. To ease the task of preserving conservation properties numerically, the equations can be manipulated into the following form in which such cancellations can be built into the discretisation:

$$\frac{\partial \tilde{g}_i}{\partial \tilde{t}} + \frac{\tilde{v}_\parallel}{\tilde{n}_i}\frac{\partial \tilde{n}_i \tilde{g}_i}{\partial \tilde{z}} - \frac{1}{2}\frac{\partial \tilde{\phi}}{\partial \tilde{z}}\frac{\partial \tilde{g}_i}{\partial \tilde{v}_\parallel} = -\tilde{R}_{\rm in}\tilde{n}_n\left(\tilde{g}_i - \tilde{g}_n\right) + \frac{\tilde{g}_i}{\tilde{n}_i}\frac{\partial \tilde{n}_i \tilde{u}_i}{\partial \tilde{z}}, \tag{30}$$

$$\frac{\partial \tilde{g}_n}{\partial \tilde{t}} + \frac{\tilde{v}_\parallel}{\tilde{n}_n}\frac{\partial \tilde{n}_n \tilde{g}_n}{\partial \tilde{z}} = -\tilde{R}_{\rm in}\tilde{n}_i\left(\tilde{g}_n - \tilde{g}_i\right) + \frac{\tilde{g}_n}{\tilde{n}_n}\frac{\partial \tilde{n}_n \tilde{u}_n}{\partial \tilde{z}}, \tag{31}$$

$$\frac{\partial \tilde{n}_s}{\partial \tilde{t}} + \frac{\partial \tilde{n}_s \tilde{u}_s}{\partial \tilde{z}} = 0, \tag{32}$$

$$e^{\tilde{\phi}} = \tilde{n}_i, \tag{33}$$

and

$$\tilde{u}_s = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\tilde{v}_\parallel \tilde{g}_n \tilde{v}_\parallel. \tag{34}$$

2.2. Moment approach: parallel flow

The parallel flow can also be evolved separately by switching from $v_\parallel$ as a coordinate to $w_\parallel \doteq v_\parallel - u$. With this change of variable, the normalised kinetic equations (30) and (31) become

$$\frac{\partial \tilde{g}_i}{\partial \tilde{t}} + \frac{\tilde{v}_\parallel}{\tilde{n}_i}\frac{\partial \tilde{n}_i \tilde{g}_i}{\partial \tilde{z}} - \left(\frac{1}{2}\frac{\partial \tilde{\phi}}{\partial \tilde{z}} + \frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_i\frac{\partial \tilde{u}_i}{\partial \tilde{z}} + \tilde{w}_\parallel\frac{\partial \tilde{u}_i}{\partial \tilde{z}}\right)\frac{\partial \tilde{g}_i}{\partial \tilde{w}_\parallel} = -\tilde{R}_{\rm in}\tilde{n}_n\left(\tilde{g}_i - \tilde{g}_n\right) + \frac{\tilde{g}_i}{\tilde{n}_i}\frac{\partial \tilde{n}_i \tilde{u}_i}{\partial \tilde{z}} \tag{35}$$

and

$$\frac{\partial \tilde{g}_n}{\partial \tilde{t}}+\frac{\tilde{v}_\parallel}{\tilde{n}_n}\frac{\partial \tilde{n}_n \tilde{g}_n}{\partial \tilde{z}}-\left(\frac{\partial \tilde{u}_n}{\partial \tilde{t}}+\tilde{u}_n\frac{\partial \tilde{u}_n}{\partial \tilde{z}}+\tilde{w}_\parallel\frac{\partial \tilde{u}_n}{\partial \tilde{z}}\right)\frac{\partial \tilde{g}_n}{\partial \tilde{w}_\parallel}=-\tilde{R}_{\rm in}\tilde{n}_i\left(\tilde{g}_n-\tilde{g}_i\right)+\frac{\tilde{g}_n}{\tilde{n}_n}\frac{\partial \tilde{n}_n\tilde{u}_n}{\partial \tilde{z}}, \quad (36)$$

with $\tilde{u}_s$ determined by the momentum equations

$$m_i n_i\left(\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial z}\right)=-\frac{\partial p_{\parallel,i}}{\partial z}-en_i\frac{\partial \phi}{\partial z}+m_i R_{\rm in} n_i n_n\left(u_n-u_i\right) \quad (37)$$

and

$$m_n n_n\left(\frac{\partial u_n}{\partial t}+u_n\frac{\partial u_n}{\partial z}\right)=-\frac{\partial p_{\parallel,n}}{\partial z}+m_n R_{\rm in} n_i n_n\left(u_i-u_n\right), \quad (38)$$

where

$$p_{\parallel,s}\doteq\int_{-\infty}^{\infty}dw_\parallel m_s w_\parallel^2 f_s=n_s T_{\parallel,s}. \quad (39)$$

Substituting the momentum equations (37) and (38) into Eqs. (35) and (36) gives

$$\frac{\partial \tilde{g}_i}{\partial \tilde{t}}+\frac{\tilde{w}_\parallel+\tilde{u}_i}{\tilde{n}_i}\frac{\partial \tilde{n}_i\tilde{g}_i}{\partial \tilde{z}}+\dot{w}_{\parallel,\rm i}\frac{\partial \tilde{g}_i}{\partial \tilde{w}_\parallel}=-\tilde{R}_{\rm in}\tilde{n}_n\left(\tilde{g}_i-\tilde{g}_n\right)+\frac{\tilde{g}_i}{\tilde{n}_i}\frac{\partial \tilde{n}_i\tilde{u}_i}{\partial \tilde{z}} \quad (40)$$

and

$$\frac{\partial \tilde{g}_n}{\partial \tilde{t}}+\frac{\tilde{w}_\parallel+\tilde{u}_n}{\tilde{n}_n}\frac{\partial \tilde{n}_n\tilde{g}_n}{\partial \tilde{z}}+\dot{w}_{\parallel,n}\frac{\partial \tilde{g}_n}{\partial \tilde{w}_\parallel}=-\tilde{R}_{\rm in}\tilde{n}_i\left(\tilde{g}_n-\tilde{g}_i\right)+\frac{\tilde{g}_n}{\tilde{n}_n}\frac{\partial \tilde{n}_n\tilde{u}_n}{\partial \tilde{z}} \quad (41)$$

with

$$\dot{w}_{\parallel,\rm i}=-\tilde{w}_\parallel\frac{\partial \tilde{u}_i}{\partial \tilde{z}}+\frac{1}{\tilde{n}_i}\frac{\partial \tilde{p}_{\parallel,i}}{\partial \tilde{z}}-\tilde{R}_{\rm in}\tilde{n}_n\left(\tilde{u}_n-\tilde{u}_i\right) \quad (42)$$

and

$$\dot{w}_{\parallel,n}=-\tilde{w}_\parallel\frac{\partial \tilde{u}_n}{\partial \tilde{z}}+\frac{1}{\tilde{n}_n}\frac{\partial \tilde{p}_{\parallel,n}}{\partial \tilde{z}}-\tilde{R}_{\rm in}\tilde{n}_i\left(\tilde{u}_i-\tilde{u}_n\right) \quad (43)$$

where

$$\tilde{p}_{\parallel,s}\doteq\frac{p_{\parallel,s}}{N_e m_s v_{\rm th,i}^2}=\frac{p_{\parallel,s}}{2N_e T_e}. \quad (44)$$

We choose to evolve the momentum equations in conservative form:

$$\frac{\partial}{\partial t}\left(m_i n_i u_i\right)=-\frac{\partial}{\partial z}\left(p_{\parallel,i}+m_i n_i u_i^2\right)-en_i\frac{\partial \phi}{\partial z}+m_i R_{\rm in} n_i n_n\left(u_n-u_i\right) \quad (45)$$

and

$$\frac{\partial}{\partial t}\left(m_n n_n u_n\right)=-\frac{\partial}{\partial z}\left(p_{\parallel,n}+m_n n_n u_n^2\right)+m_n R_{\rm in} n_i n_n\left(u_i-u_n\right), \quad (46)$$

which, when normalised, become

$$\frac{\partial}{\partial t}\left(\tilde{n}_i\tilde{u}_i\right)=-\frac{\partial}{\partial \tilde{z}}\left(\tilde{p}_{\parallel,i}+\tilde{n}_i\tilde{u}_i^2\right)-\frac{\tilde{n}_i}{2}\frac{\partial \tilde{\phi}}{\partial \tilde{z}}+\tilde{R}_{\rm in}\tilde{n}_i\tilde{n}_n\left(\tilde{u}_n-\tilde{u}_i\right) \quad (47)$$

and

$$\frac{\partial}{\partial t}\left(\tilde{n}_n\tilde{u}_n\right)=-\frac{\partial}{\partial \tilde{z}}\left(\tilde{p}_{\parallel,n}+\tilde{n}_n\tilde{u}_n^2\right)+\tilde{R}_{\rm in}\tilde{n}_i\tilde{n}_n\left(\tilde{u}_i-\tilde{u}_n\right). \quad (48)$$

2.3. Moment approach: parallel pressure

We additionally separate the parallel pressure (or, equivalently, the thermal speed) by changing variables from the peculiar velocity to the modified peculiar velocity $w_\|$, given by

$$w_\| \doteq \frac{v_\| - u_s}{v_{\mathrm{th},s}}, \tag{49}$$

where $v_{\mathrm{th},s} = \sqrt{2T_{\|,s}/m_s}$. Note that $T_{\|,s}$ is related to the distribution function via the equation of state,

$$T_{\|,s} = \frac{p_{\|,s}}{n_s} \doteq \frac{m_s v_{\mathrm{th},s}^2}{2} \int dw_\| 2w_\|^2 \frac{v_{\mathrm{th},s} f_s}{n_s} \Rightarrow \int dw_\| w_\|^2 \frac{v_{\mathrm{th},s} f_s}{n_s} = \frac{1}{2}, \tag{50}$$

and the other relevant moments of the distribution function are expressed in terms of the modified peculiar velocity as

$$n_s = \int dw_\| v_{\mathrm{th},s} f_s, \tag{51}$$

$$n_s u_s = \int dw_\| w_\| v_{\mathrm{th},s}^2 f_s, \tag{52}$$

$$p_{\|,s} = \int dw_\| m_s w_\|^2 v_{\mathrm{th},s}^3 f_s, \tag{53}$$

and

$$q_{\|,s} = \int dw_\| m_s w_\|^3 v_{\mathrm{th},s}^4 f_s. \tag{54}$$

In terms of the coordinates $(z, w_\|)$, the drift kinetic equations for the ions and neutrals are

$$\frac{\partial f_s}{\partial t} + \left(v_{\mathrm{th},s} w_\| + u_s\right) \frac{\partial f_s}{\partial z} + \dot{w}_{\|,s} \frac{\partial f_s}{\partial w_\|} = -R_{ss'} \left(n_{s'} f_s - n_s f_{s'}\right), \tag{55}$$

where

$$\begin{aligned} \dot{w}_{\|,s} = &- w_\|^2 \frac{\partial v_{\mathrm{th},s}}{\partial z} + \frac{1}{v_{\mathrm{th},s}} \left(\frac{1}{m_s n_s} \frac{\partial p_{\|,s}}{\partial z} - R_{\mathrm{ss'}} n_{s'} \left(u_{s'} - u_s\right) \right) \\ &+ \frac{w_\|}{m_s n_s v_{\mathrm{th},s}^2} \left(\frac{\partial q_{\|,s}}{\partial z} + R_{\mathrm{ss'}} \left(n_{s'} p_{\|,s} - n_s p_{\|,s'}\right) \right) \end{aligned} \tag{56}$$

and $(s, s') = (\mathrm{i}, n)$ or $(s, s') = (n, \mathrm{i})$. To obtain this result, we made use of the momentum equations (37)-(38) and the energy equation,

$$\begin{aligned} \frac{\partial p_{\|,s}}{\partial t} + u_s \frac{\partial p_{\|,s}}{\partial z} &= m_s n_s v_{\mathrm{th},s} \left(\frac{\partial v_{\mathrm{th},s}}{\partial t} + u_s \frac{\partial v_{\mathrm{th},s}}{\partial z} \right) + T_{\|,s} \left(\frac{\partial n_s}{\partial t} + u_s \frac{\partial n_s}{\partial z} \right) \\ &= -\frac{\partial q_{\|,s}}{\partial z} - 3p_{\|,s} \frac{\partial u_s}{\partial z} - R_{\mathrm{ss'}} \left(n_{s'} p_{\|,s} - n_s p_{\|,s'}\right), \end{aligned} \tag{57}$$

recognising that $p_{\|,s} = m_s n_s v_{\mathrm{th},s}^2/2$.

We next define a modified distribution function that allows for separate treatment of the density and parallel pressure:

$$g_s \doteq \frac{v_{\mathrm{th},s}}{n_s} f_s = \left(\frac{2}{m_s}\right)^{1/2} \frac{p_{\parallel,s}^{1/2}}{n_s^{3/2}} f_s. \tag{58}$$

In terms of g_s the drift kinetic equation is

$$\frac{\partial g_s}{\partial t} + \frac{g_s}{n_s}\frac{\partial n_s}{\partial t} - \frac{g_s}{v_{\mathrm{th},s}}\frac{\partial v_{\mathrm{th},s}}{\partial t} + \frac{v_{\mathrm{th},s}}{n_s}\left(v_{\mathrm{th},s} w_\parallel + u_s\right)\frac{\partial f_s}{\partial z} + \dot{w}_{\parallel,s}\frac{\partial g_s}{\partial w_\parallel} = -R_{ss'} n_{s'}\left(g_s - \frac{v_{\mathrm{th},s}}{v_{\mathrm{th},s'}} g_{s'}\right). \tag{59}$$

Substituting the continuity and energy equations to eliminate $\partial n_s/\partial t$ and $\partial v_{\mathrm{th},s}/\partial t$ gives

$$\begin{aligned} &\frac{\partial g_s}{\partial t} + \frac{v_{\mathrm{th},s}}{n_s}\left(v_{\mathrm{th},s} w_\parallel + u_s\right)\frac{\partial f_s}{\partial z} + \dot{w}_{\parallel,s}\frac{\partial g_s}{\partial w_\parallel} + R_{ss'} n_{s'}\left(g_s - \frac{v_{\mathrm{th},s}}{v_{\mathrm{th},s'}} g_{s'}\right) \\ &\quad = \frac{g_s u_s}{n_s}\frac{\partial n_s}{\partial z} - \frac{g_s}{v_{\mathrm{th},s}}\left(u_s \frac{\partial v_{\mathrm{th},s}}{\partial z} + \frac{1}{m_s n_s v_{\mathrm{th},s}}\frac{\partial q_{\parallel,s}}{\partial z} + \frac{R_{\mathrm{ss'}}}{m_s n_s v_{\mathrm{th},s}}\left(n_{s'} p_{\parallel,s} - n_s p_{\parallel,s'}\right)\right). \end{aligned} \tag{60}$$

Finally, we normalise the various equations. The normalised distribution function is

$$\tilde{f}_s \doteq f_s \frac{c_s\sqrt{\pi}}{N_e} = \tilde{g}_s \frac{\tilde{n}_s}{\tilde{v}_{\mathrm{th},s}}, \tag{61}$$

where $\tilde{g}_s = g_s\sqrt{\pi}$ and $\tilde{v}_{\mathrm{th},s} = v_{\mathrm{th},s}/c_s$. The drift kinetic equation is normalised by multiplying each term by $\sqrt{\pi}L_z/c_s$:

$$\begin{aligned} &\frac{\partial \tilde{g}_s}{\partial \tilde{t}} + \frac{\tilde{v}_{\mathrm{th},s}}{\tilde{n}_s}\left(\tilde{v}_{\mathrm{th},s} w_\parallel + \tilde{u}_s\right)\frac{\partial \tilde{f}_s}{\partial \tilde{z}} + \tilde{\dot{w}}_{\parallel,s}\frac{\partial \tilde{g}_s}{\partial w_\parallel} + \tilde{R}_{\mathrm{ss'}}\tilde{n}'_s\left(\tilde{g}_s - \frac{v_{\mathrm{th},s}}{v_{\mathrm{th},s'}}\tilde{g}_{s'}\right) \\ &\quad = \frac{\tilde{g}_s \tilde{u}_s}{\tilde{n}_s}\frac{\partial \tilde{n}_s}{\partial \tilde{z}} - \tilde{g}_s\left(\frac{\tilde{u}_s}{\tilde{v}_{\mathrm{th},s}}\frac{\partial \tilde{v}_{\mathrm{th},s}}{\partial \tilde{z}} + \frac{1}{2\tilde{p}_{\parallel,s}}\frac{\partial \tilde{q}_{\parallel,s}}{\partial \tilde{z}} + \frac{\tilde{R}_{\mathrm{ss'}}}{2\tilde{p}_{\parallel,s}}\left(\tilde{n}'_s\tilde{p}_{\parallel,s} - \tilde{n}_s\tilde{p}_{\parallel,s'}\right)\right), \end{aligned} \tag{62}$$

where

$$\tilde{\dot{w}}_{\parallel,s} = -w_\parallel^2\frac{\partial \tilde{v}_{\mathrm{th},s}}{\partial \tilde{z}} + \frac{1}{\tilde{v}_{\mathrm{th},s}}\left(\frac{1}{\tilde{n}_s}\frac{\partial \tilde{p}_{\parallel,s}}{\partial \tilde{z}} - \tilde{R}_{\mathrm{ss'}}\tilde{n}'_s\left(\tilde{u}_{s'} - \tilde{u}_s\right)\right) + \frac{w_\parallel}{2\tilde{p}_{\parallel,s}}\left(\frac{\partial \tilde{q}_{\parallel,s}}{\partial z} + \tilde{R}_{\mathrm{ss'}}\left(\tilde{n}'_s\tilde{p}_{\parallel,s} - \tilde{n}_s\tilde{p}_{\parallel,s'}\right)\right), \tag{63}$$

$$\tilde{n}_s \doteq \frac{n_s}{N_e} = \frac{\tilde{v}_{\mathrm{th},s}}{\sqrt{\pi}}\int dw_\parallel \tilde{f}_s, \tag{64}$$

$$\tilde{n}_s\tilde{u}_s \doteq \frac{n_s}{N_e}\frac{u_s}{c_s} = \frac{\tilde{v}_{\mathrm{th},s}^2}{\sqrt{\pi}}\int dw_\parallel w_\parallel \tilde{f}_s \tag{65}$$

$$\tilde{p}_{\parallel,s} \doteq \frac{p_{\parallel,s}}{m_s N_e c_s^2} = \frac{\tilde{v}_{\mathrm{th},s}^3}{\sqrt{\pi}}\int dw_\parallel w_\parallel^2 \tilde{f}_s, \tag{66}$$

$$\tilde{v}_{\mathrm{th},s} \doteq \frac{v_{\mathrm{th},s}}{c_s} = \sqrt{\frac{2\tilde{p}_{\parallel,s}}{\tilde{n}_s}} \tag{67}$$

and

$$\tilde{q}_{\parallel,s} = \frac{q_{\parallel,s}}{m_s N_e c_s^3} = \frac{\tilde{v}_{\mathrm{th},s}^4}{\sqrt{\pi}} \int d\tilde{w}_\parallel \tilde{w}_\parallel^3 \tilde{f}_s. \tag{68}$$

The energy equation is normalised by multiplying through by $L_z/m_s N_e c_s^3$:

$$\frac{\partial \tilde{p}_{\parallel,s}}{\partial \tilde{t}} + \tilde{u}_s \frac{\partial \tilde{p}_{\parallel,s}}{\partial \tilde{z}} + \frac{\partial \tilde{q}_{\parallel,s}}{\partial \tilde{z}} + 3\tilde{p}_{\parallel,s} \frac{\partial \tilde{u}_s}{\partial \tilde{z}} + \tilde{R}_{\mathrm{ss'}} \left(\tilde{n}'_s \tilde{p}_{\parallel,s} - \tilde{n}_s \tilde{p}_{\parallel,s'} \right) = 0. \tag{69}$$

Note that in these coordinates the normalised particle distribution function should satisfy the following properties related to particle number, momentum and energy conservation:

$$\frac{1}{\sqrt{\pi}} \int dw_\parallel \tilde{g}_s = 1, \tag{70}$$

$$\frac{1}{\sqrt{\pi}} \int dw_\parallel w_\parallel \tilde{g}_s = 0, \tag{71}$$

and

$$\frac{1}{\sqrt{\pi}} \int dw_\parallel w_\parallel^2 \tilde{g}_s = \frac{1}{2} \tag{72}$$

3. Numerical implementation

A detailed description of the time and space discretisation employed in the code is given in [2], and the code itself is publicly available at `https://github.com/mabarnes/moment_kinetics`. Here we focus on a novel extension to our algorithm that is necessary to ensure exact numerical satisfaction of the conservation properties (70)-(72).

The currently-favoured approach in the code for satisfying exactly the desired conservation properties is to correct the numerical solutions for n and g at the end of each time step. For the density, one can set

$$n^{m+1} = \hat{n}^{m+1} + n^m \left(1 - \frac{\int dz\ \hat{n}^{m+1}}{\int dz\ n^m} \right), \tag{73}$$

where $\hat{n}^{m+1}$ is the updated solution (at time level $m+1$) to the continuity equation before applying any conserving correction. This guarantees that $\int dz\, (n^{m+1} - n^m) = 0$. Note that the superscripts here refer to the time level, not the element index. The additional error in the density introduced by this correction is

$$\begin{aligned} n^m \left(1 - \frac{\int dz\ \hat{n}^{m+1}}{\int dz\ n^m} \right) &= n^m \left(1 - \frac{\int dz \left(n_{\mathrm{exact}}^{m+1} + \epsilon^m \right)}{\int dz\ n^m} \right) \\ &= n^m \frac{\int dz\ \epsilon^m}{\int dz\ n^m} = \mathcal{O}(\epsilon^m), \end{aligned} \tag{74}$$

where ϵ^m is the error due to numerical discretisation, and n_{exact}^{m+1} is the solution for $\hat{n}^{m+1}$ in the limit $\epsilon^m = 0$.

A similar technique can be applied to conserve $\int dw_\parallel g = 1$, $\int dw_\parallel w_\parallel g = 0$ and $\int dw_\parallel w_\parallel^2 g = 1/2$, where here we are focusing on the case in which the density, parallel flow and parallel pressure are all evolved separately. In particular, we set

$$\begin{aligned} g^{m+1} = \hat{g}^{m+1} + g^m \left(1 - \int dw_\parallel \hat{g}^{m+1}\right) - w_\parallel g_E^m \frac{\int dw_\parallel w_\parallel \hat{g}^{m+1}}{\int dw_\parallel w_\parallel^2 g_E^m} \\ - \left(w_\parallel^2 - \frac{1}{2}\right) g_E^m \frac{\int dw_\parallel \left(w_\parallel^2 - 1/2\right) \hat{g}^{m+1}}{\int dw_\parallel w_\parallel^2 \left(w_\parallel^2 - 1/2\right) g_E^m}, \end{aligned} \tag{75}$$

where $\hat{g}^{m+1}$ is the updated solution to the drift kinetic equation before applying any conserving correction and $g_E(w_\parallel) = (g(w_\parallel) + g(-w_\parallel))/2$ is the even-in-$w_\parallel$ component of g. Again, the additional error in g associated with this correction is $\mathcal{O}(\delta^m)$, where δ^m is the discretisation error. The correction ensures that $\int dw_\parallel g^{m+1} = 1$, $\int dw_\parallel w_\parallel g^{m+1} = 0$ and $\int dw_\parallel w_\parallel^2 g^{m+1} = 1/2$, provided the corresponding properties are satisfied for g^m.

It is thus critical to carefully choose the initial conditions in the code so that these properties are initially satisfied to machine precision. To do this we first set initial conditions on the density, parallel flow and parallel pressure profiles, and then set the initial, normalised distribution function, $\hat{g}^0$, to be an even function of $w_\parallel$. The constraint that $\hat{g}^0$ be even is not necessary, but is currently chosen for convenience as it automatically ensures that $\int dw_\parallel w_\parallel \hat{g}^0 = 0$. This initial distribution function is then corrected in a manner analogous to $\hat{g}^{m+1}$ above:

$$\begin{aligned} g^0 = \frac{\hat{g}^0}{\int dw_\parallel \hat{g}^0} + \left(\frac{1}{2} - \frac{\int dw_\parallel w_\parallel^2 \hat{g}^0}{\int dw_\parallel \hat{g}^0}\right) \left(\frac{w_\parallel^2 \hat{g}^0}{\int dw_\parallel w_\parallel^2 \hat{g}^0} - \frac{\hat{g}^0}{\int dw_\parallel \hat{g}^0}\right) \\ \Bigg/ \left(\int dw_\parallel w_\parallel^2 \left(\frac{w_\parallel^2 \hat{g}^0}{\int dw_\parallel w_\parallel^2 \hat{g}^0} - \frac{\hat{g}^0}{\int dw_\parallel \hat{g}^0}\right)\right) \end{aligned} \tag{76}$$

This approach is simple, does not change the order of accuracy of the discretisation scheme and allows for the use of numerical dissipation to improve numerical stability properties. Results showing its efficacy are given in Sec. 4

4. Numerical results

To benchmark our numerical implementation of the moment-based approach encapsulated in Eqs. (62)-(69), we compare our simulation results with the analytical benchmarks developed in [3] and with the numerical results obtained by directly solving the kinetic system corresponding to Eqs. (12)-(15). We note that the results obtained with separate evolution of only the density (Eqs. (30)-(34)) and of only the density and parallel flow (Eqs. (40)-(48)) are almost identical to the ones presented here in which all three of the lowest-order moments are evolved separately. The results reported here were obtained using the conserving corrections given by Eqs. 73 and 75.

We have initialised the modified distribution functions for the ions and neutrals to be of the form

$$\tilde{g}_s = \exp\left(-w_\parallel^2\right). \tag{77}$$

The initial temperature T_s is chosen to be T_e, which is constant along z, and the initial density is chosen to be $\tilde{n}_s = \overline{n}_s + \delta n_s$, with $\overline{n}_s = \int dz(n_s/N_e)/L_z$ the field-line-averaged density, normalised to the z-independent electron density N_e. The piece of the density that varies along z, δn_s, is chosen to be small compared to $\overline{n}_s$ ($\delta n_s/\overline{n}_s = 0.001$) so that the system of equations can be linearised to a good approximation. This facilitates comparisons with the linear analytical theory presented in [3]. For all cases shown here, $\overline{n}_i = \overline{n}_n = N_e/2$, $\overline{T}_i = \overline{T}_n = T_e$ and $m_i = m_n$. The charge exchange collision frequency is varied, and damping rates and frequencies are extracted by considering the time evolution of the spatially-varying component of the electrostatic potential, $\delta\phi$. In particular, a least-squares fit for $\delta\phi(t)/\delta\phi(t_0)$ is done for each simulation to a function of the form $\exp(-\gamma(t-t_0))\cos(\omega t - \varphi)/\cos(\omega t_0 - \varphi)$ to obtain the damping rate γ, frequency ω and phase φ. The results are given in Fig. 1. There is good agreement across a wide range of charge exchange collision frequencies, both for the damping of finite frequency modes (corresponding to the solid lines) and to a zero frequency mode that appears at larger collisionalities (dashed-dotted lines). The minor discrepancy between the analytical and numerical damping rates that is apparent for the case with normalised charge exchange collision frequency near 0.7 is due to the simultaneous presence of both modes with similar damping rates.

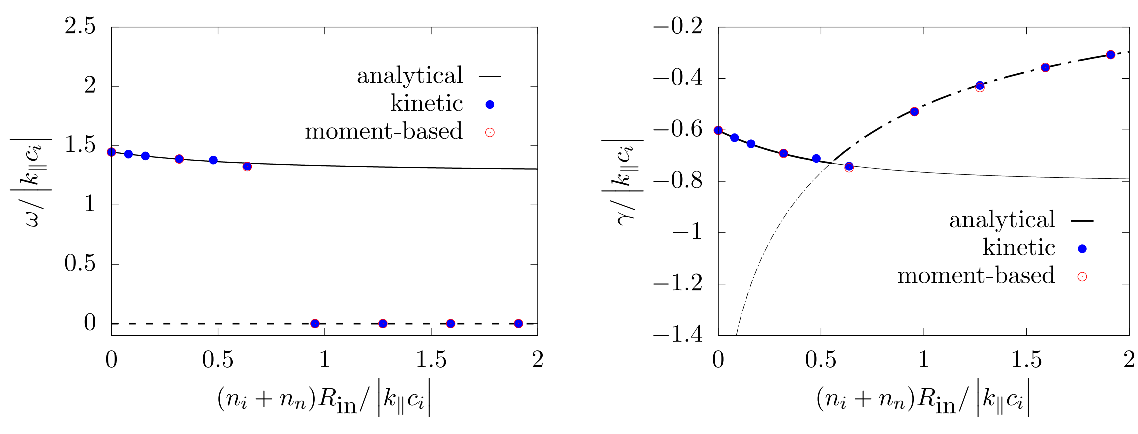

Figure 1. Normalized damping rate and real frequency as a function of the charge exchange collision frequency.

In Figure 2 we show the difference in conservation properties between cases for which the conservative corrections indicated at the beginning of the Section are employed and those for which no conserving correction is applied. With the conservative implementation, all of the requisite moments of the modified distribution function are conserved to machine precision, regardless of numerical resolution.

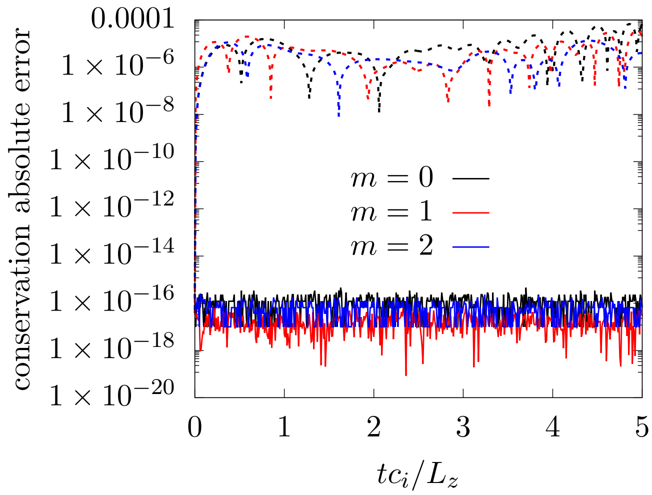

Figure 2. Time traces of the deviation from exact conservation of the moments $\int dw_\parallel w_\parallel^m g$ for low-resolution ($N_z = 9$ on one element, $N_v = 9$ on five elements) simulations with normalised $R_{\rm in} \approx 0.3$. Solid and dotted lines correspond to simulations with and without conserving corrections, respectively.

5. Future plans

Now that we have a proof-of-concept implementation of the moment-based approach to solving the 1+1D kinetic problem with periodic boundary conditions, we plan to extend the model to treat wall boundary conditions.

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