2D drift kinetic model with wall boundary conditions

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1. Introduction

In previous reports, we proposed 1D drift kinetic equations with periodic boundary conditions, adequate for the closed field line region of the edge, and wall boundary conditions. In this report, we build a 2D drift kinetic model for a helical magnetic field. The helical magnetic field has similarities with the magnetic field in the tokamak edge.

2. Helical magnetic field

To describe the geometry of the magnetic field, we use the cylindrical coordinates $\{r, z, \zeta\}$ (see figure 1). In this coordinates, the helical field is

$$\mathbf{B}(r, \zeta) := B_z(r)\hat{\mathbf{z}} + B_\zeta(r)\hat{\boldsymbol{\zeta}}(\zeta), \tag{2.1}$$

where $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\zeta}}$ are the unit vectors in the direction of ∇z and $\nabla \zeta$. Note that the components B_z and B_ζ only depend on the radial position r.

In principle, one can use any $B_z(r)$ and $B_\zeta(r)$. There is a particular choice that is more physical. In the edge, the magnetic field is determined by currents running through the core plasma or through external magnets. Thus, according to Ampére's law, the magnetic field in the edge should satisfy $\nabla \times \mathbf{B} \simeq 0$. This condition imposes that B_z be a constant and that B_ζ decay as $1/r$,

$$B_\zeta(r) = \frac{I}{r}, \tag{2.2}$$

where I is a constant determined by the vertical current through the core plasma.

3. Geometry and orderings

We consider a magnetized plasma with one ion species with charge e and mass m_i, electrons with charge $-e$ and mass m_e, and one species of neutrals with mass

$$m_n = m_i. \tag{3.1}$$

The plasma is magnetized by a helical magnetic field like the one described in the previous section, and we assume that the plasma only varies in r and z. We assume that the electric field produced by the plasma is electrostatic, $\mathbf{E} = -(\partial\phi/\partial r)\hat{\mathbf{r}} - (\partial\phi/\partial z)\hat{\mathbf{z}}$, where $\hat{\mathbf{r}}$ is the unit vector in the direction ∇r. The potential $\phi(r, z, t)$ depends on the coordinates r and z and on time t.

There are two conducting walls at $z = 0$ and $z = L_z$. In the radial direction, we consider the interval between $r = r_0$ and $r = r_0 + L_r$. The length L_r is determined by a balance between the fast parallel velocity of the particles along magnetic field lines

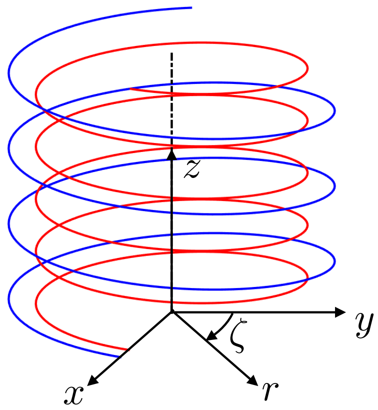

FIGURE 1. Two magnetic field lines (in blue and red) of the helical magnetic field. Note that the direction of the angle ζ is opposite to the direction usually chosen in cylindrical coordinates.

and their slow drift across them. The characteristic length between the two walls along a magnetic field line is of order

$$L_\parallel \sim \frac{B}{B_z} L_z. \tag{3.2}$$

Thus, the typical time that it takes an ion to move from wall to wall is $L_\parallel / v_{ti} \sim (B/B_z)(L_z/v_{ti})$, where $v_{ti} := \sqrt{2T_i/m_i}$ is the ion thermal speed and T_i is the ion temperature. For a potential ϕ of the order of T_i/e, where e is the proton charge, the radial $\mathbf{E} \times \mathbf{B}$ drift is

$$v_{Er} := -\frac{B_\zeta}{B^2} \frac{\partial \phi}{\partial z} \sim \frac{\rho_i}{L_z} v_{ti}, \tag{3.3}$$

where $\rho_i := v_{ti}/\Omega_i$ is the characteristic ion gyroradius and $\Omega_i := eB/m_i$ is the ion gyrofrequency. Thus, the time it takes for an ion to cross the domain in the radial direction is $L_r/v_{Er} \sim L_r L_z / \rho_i v_{ti}$. By making this time of the same order as $L_\parallel / v_{ti}$, we solve for L_r to find

$$L_r \sim \frac{B}{B_z} \rho_i \tag{3.4}$$

To simplify the problem to a tractable drift kinetic form, we assume that ρ_i is much smaller than L_r. This implies that

$$\frac{\rho_i}{L_r} \sim \frac{B_z}{B} \sim \frac{B_z}{B_\zeta} \ll 1, \tag{3.5}$$

that is, we will limit our model to magnetic fields that are mostly azimuthal and have a very small vertical component. This is an approximation that is consistent with magnetic field geometry in conventional tokamaks and also in the edge of many shots in spherical tokamaks.

We also assume that $r_0 \sim L_z \gg L_r$. Since r_0 is the characteristic length of variation of the magnetic field $\mathbf{B}$, the magnetic field barely changes across the domain $[r_0, r_0 + L_r]$. Thus, within our ordering, we assume $\mathbf{B}$ to be uniform in the domain of interest.

Our orderings above rest on the assumption $\phi \sim T_i/e$. This ordering is a result of the wall boundary conditions that impose $\phi \sim T_e/e$ (see section 5) and the fact that $T_i \sim T_e$ due to collisional temperature equilibration.

4. 2D electrostatic drift kinetics

We assume that the gyroradii are small compared to the length scales of interest, and that the gyrofrequencies are much larger than the frequencies that we want to model. Thus, the distribution functions $f_s(r, z, v_\parallel, v_\perp, t)$ of the charged species $s = i, e$ only depend on the component of the velocity parallel to the magnetic field $v_\parallel$ and the magnitude of the velocity perpendicular to the magnetic field $v_\perp$, and are independent of the direction of the velocity perpendicular to the magnetic field (Hazeltine 1973). The distribution functions of ions and electrons ($s = i, e$) that in general can depend on three spatial variables $\mathbf{r}$, three components of the velocity $\mathbf{v}$ and the time t depend only on r, z, $v_\parallel$, $v_\perp$ and t,

$$f_s(\mathbf{r}, \mathbf{v}, t) = f_s(r, z, v_\parallel, v_\perp, t). \tag{4.1}$$

The neutral distribution function depends in general on the three velocity components,

$$f_n(\mathbf{r}, \mathbf{v}, t) = f_n(r, z, v_r, v_z, v_\zeta, t). \tag{4.2}$$

We remind the reader that the model is 2D because we have assumed that the plasma parameters do not depend on the angle ζ.

The equations for the distribution functions of the different species are

$$\begin{aligned}\frac{\partial f_i}{\partial t} - \frac{1}{B}\frac{\partial \phi}{\partial z}\frac{\partial f_i}{\partial r} + \left(\frac{v_\parallel B_z}{B} + \frac{1}{B}\frac{\partial \phi}{\partial r}\right)\frac{\partial f_i}{\partial z} - \frac{eB_z}{m_i B}\frac{\partial \phi}{\partial z}\frac{\partial f_i}{\partial v_\parallel} \\ = C_{ii}[f_i] + \langle C_{in}[f_i, f_n]\rangle + \langle C_{i,\mathrm{ion}}[f_e, f_n]\rangle + C_{ie}[f_i, f_e] + S_i,\end{aligned} \tag{4.3}$$

$$\begin{aligned}\frac{\partial f_e}{\partial t} - \frac{1}{B}\frac{\partial \phi}{\partial z}\frac{\partial f_e}{\partial r} + \left(\frac{v_\parallel B_z}{B} + \frac{1}{B}\frac{\partial \phi}{\partial r}\right)\frac{\partial f_e}{\partial z} + \frac{eB_z}{m_e B}\frac{\partial \phi}{\partial z}\frac{\partial f_e}{\partial v_\parallel} = C_{ee}[f_e] \\ + C_{ei}[f_e, f_i]\left[1 + O\left(\frac{m_e}{m_i}\right)\right] + \left\langle C_{en}[f_e, f_n]\left[1 + O\left(\frac{m_e}{m_i}\right)\right]\right\rangle + \langle C_{e,\mathrm{ion}}[f_e, f_n]\rangle + S_e\end{aligned} \tag{4.4}$$

and

$$\frac{\partial f_n}{\partial t} + v_r\frac{\partial f_n}{\partial r} + v_z\frac{\partial f_n}{\partial z} = C_{ni}[f_n, f_i] + C_{ne}[f_n, f_e] + C_{n,\mathrm{ion}}[f_n, f_e] + S_n. \tag{4.5}$$

The triangular brackets on a function $G(r, z, v_r, v_z, v_\zeta, t)$ indicate gyoraverage,

$$\langle G\rangle(r, z, v_\parallel, v_\perp, t) := \frac{1}{2\pi}\int_0^{2\pi} G\left(r, z, v_\perp\cos\varphi, v_\perp\sin\varphi, v_\parallel, t\right)\,\mathrm{d}\varphi. \tag{4.6}$$

The sources $S_s(r, z, \mathbf{v}, t)$ with $s = i, e, n$ represent heating, fueling and the effect of turbulence. The ion and electron particle sources satisfy

$$\int S_i\,\mathrm{d}^3v = \int S_e\,\mathrm{d}^3v. \tag{4.7}$$

Note that we have neglected the curvature and ∇B drifts in equations (4.3) and (4.4). These drifts point in the z- and ζ-direction. The ζ-direction is unimportant because it is a direction of symmetry, whereas in the z-direction, the magnetic drifts can be neglected compared to the much larger terms due to the parallel velocity, $v_\parallel B_z/B$, and the z-component of the $\mathbf{E}\times\mathbf{B}$ drift, $v_{Ez} \simeq B^{-1}(\partial\phi/\partial r)$.

We have included the following collisions.

- Ion-ion and electron-electron collisions are modeled by the Fokker-Planck collision operators $C_{ii}[f_i]$ and $C_{ee}[f_e]$ (Rosenbluth *et al.* 1957),

$$C_{ss}[f_s] := \frac{2\pi e^4 \ln\Lambda}{(4\pi\epsilon_0)^2 m_s^2} \nabla_v \cdot \left(\mathbf{D}[f_s]\cdot\nabla_v f_s + \mathbf{P}[f_s] f_s\right), \tag{4.8}$$

where the matrix $\mathbf{D}$ is

$$\mathbf{D}[f_s] := \int \frac{|\mathbf{v}-\mathbf{v}'|^2\mathbf{I} - (\mathbf{v}-\mathbf{v}')(\mathbf{v}-\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|^3} f_s(\mathbf{v}')\,\mathrm{d}^3 v' \tag{4.9}$$

and the vector $\mathbf{P}$ is

$$\mathbf{P}[f_s] := -2\int \frac{\mathbf{v}-\mathbf{v}'}{|\mathbf{v}-\mathbf{v}'|^3} f_s(\mathbf{v}')\,\mathrm{d}^3 v'. \tag{4.10}$$

Here, $\mathbf{I}$ is the 3D unit matrix, ϵ_0 the vacuum permittivity and $\ln\Lambda \approx 15$ the Coulomb logarithm.

• The effect of electron-ion and elastic electron-neutral collisions on the electron distribution function can be simplified in the limit of small electron-ion mass ratio, $m_e/m_i \ll 1$. With this expansion, we find the simplified Fokker-Planck collision operator

$$C_{ei}[f_e, f_i] := \frac{2\pi e^4 n_i \ln\Lambda}{(4\pi\epsilon_0)^2 m_e^2} \nabla_v \cdot \left[\frac{|\mathbf{v}-\mathbf{u}_i|^2\mathbf{I} - (\mathbf{v}-\mathbf{u}_i)(\mathbf{v}-\mathbf{u}_i)}{|\mathbf{v}-\mathbf{u}_i|^3}\cdot\nabla_v f_e\right] \tag{4.11}$$

for electron-ion collisions (Braginskii 1958), and the simplified Boltzmann collision operator

$$C_{en}[f_e, f_n] := \frac{n_n}{4\pi}\int_0^\pi \mathrm{d}\chi \int_0^{2\pi} \mathrm{d}\varphi\, \sin\chi R_{en}(|\mathbf{v}-\mathbf{u}_n|,\chi)\left[f_e(\overline{\mathbf{v}}(\mathbf{v},\chi,\varphi,\mathbf{u}_n)) - f_e(\mathbf{v})\right] \tag{4.12}$$

for electron-neutral collisions. Here

$$n_s(r,z,t) := \int f_s(r,z,\mathbf{v},t)\,\mathrm{d}^3 v. \tag{4.13}$$

is the density of species s,

$$\mathbf{u}_s(r,z,t) := \frac{1}{n_s}\int \mathbf{v} f_s(r,z,\mathbf{v},t)\,\mathrm{d}^3 v \tag{4.14}$$

is the average velocity of species s,

$$\overline{\mathbf{v}}(\mathbf{v},\chi,\varphi,\mathbf{u}_n) := \mathbf{u}_n + \cos\chi(\mathbf{v}-\mathbf{u}_n) + |\mathbf{v}-\mathbf{u}_n|\sin\chi(\cos\varphi\,\hat{\mathbf{e}}_1 + \sin\varphi\,\hat{\mathbf{e}}_2) \tag{4.15}$$

is a rotation of the vector $\mathbf{v}$ centered around $\mathbf{u}_n$, $R_{en}(|\mathbf{v}-\mathbf{u}_n|,\chi)$ is a function determined by the physics of the electron-neutral collisions, and the unit vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are chosen to form an orthonormal basis with the vector $(\mathbf{v}-\mathbf{u}_n)/|\mathbf{v}-\mathbf{u}_n|$. In equation (4.4), we have indicated that both C_{ei} and C_{en} are missing pieces small in m_e/m_i. These pieces can become important because they represent collisional energy exchange and collisional heating, but they are cumbersome. We showed in report 2047357-TN-05-01 M1.3 that the moment method that we use allows us to keep these important effects in the moment equations even with the simplified collision operators (4.11) and (4.12).

• The expansion in electron-ion mass ratio also implies electron-ion collisions and electron-neutral collisions have a very small effect on f_i and f_n – the terms C_{ie} and C_{ne} in equations (4.3) and (4.5) are small compared with C_{ii} and C_{ni} by a factor of $\sqrt{m_e/m_i} \ll 1$,

$$C_{ie} \sim \sqrt{\frac{m_e}{m_i}} C_{ii}, \quad C_{ne} \sim \sqrt{\frac{m_e}{m_i}} C_{ni}. \tag{4.16}$$

Like the mass ratio corrections to C_{ei} and C_{en}, these terms can become important because they contain the collisional energy exchange between electrons and the heavier species. We will keep these effects in a simplified form in our moment formulation.

- Charge-exchange collisions are represented by the simplified Boltzmann collision operators

$$C_{in}[f_i, f_n] := -\int R_{in}(|\mathbf{v}-\mathbf{v}'|)\,[f_i(\mathbf{v})f_n(\mathbf{v}') - f_i(\mathbf{v}')f_n(\mathbf{v})]\;\mathrm{d}^3v' \tag{4.17}$$

and

$$C_{ni}[f_n, f_i] := -\int R_{in}(|\mathbf{v}-\mathbf{v}'|)\,[f_n(\mathbf{v})f_i(\mathbf{v}') - f_n(\mathbf{v}')f_i(\mathbf{v})]\;\mathrm{d}^3v'. \tag{4.18}$$

- To model ionization, we use the collision operators

$$C_{i,\mathrm{ion}}[f_e, f_n] := f_n \int R_{\mathrm{ion}}(v')f_e(\mathbf{v}')\,\mathrm{d}^3v' \tag{4.19}$$

and

$$C_{n,\mathrm{ion}}[f_e, f_n] := -f_n \int R_{\mathrm{ion}}(v')f_e(\mathbf{v}')\,\mathrm{d}^3v'. \tag{4.20}$$

We also need to include a collision operator $C_{e,\mathrm{ion}}$ in the electron equation to model the increase in the number of electrons and the energy loss due to ionization. This operator is complicated because it involves three particles (the resulting ion and two electrons), but we will be able to avoid giving it a definite form. Instead, we will use the expansion in $m_e/m_i \ll 1$ and the fact that

$$C_{e,\mathrm{ion}}[f_e, f_n] \sim n_n R_{\mathrm{ion}} f_e. \tag{4.21}$$

See report 2047357-TN-05-01 M1.3 for more details.

- We have neglected neutral-neutral collisions because, in current fusion devices, the neutral density is sufficiently small that the neutral-neutral collisions are rare.

To simplify our equations, we assume that the functions R_{en}, R_{in} and R_{ion} are constant (Connor 1977; Hazeltine *et al.* 1992; Catto 1994), finding

$$\begin{aligned}\langle C_{en}[f_e, f_n]\rangle = &- n_n R_{en}\Bigg[f_e(r, z, v_\parallel, v_\perp, t)\\ &- \frac{1}{8\pi^2}\int_0^{\pi}\mathrm{d}\chi\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\varphi'\,\sin\chi f_e\left(r, z, \overline{v}_\parallel, \overline{v}_\perp, t\right)\Bigg],\end{aligned} \tag{4.22}$$

with

$$\overline{v}_\parallel(v_\parallel, v_\perp, \chi, \varphi', \mathbf{u}_n) := u_{n\parallel} + \overline{v}_{en}\cos\chi, \tag{4.23}$$

$$\overline{v}_\perp(v_\parallel, v_\perp, \chi, \varphi, \varphi', \mathbf{u}_n) := \sqrt{u_{n\perp}^2 + \overline{v}_{en}^2\sin^2\chi - 2u_{n\perp}\overline{v}_{en}\sin\chi\cos\varphi} \tag{4.24}$$

and

$$\overline{v}_{en}(v_\parallel, v_\perp, \varphi', \mathbf{u}_n) := \sqrt{(v_\parallel - u_{n\parallel})^2 + v_\perp^2 + u_{n\perp}^2 - 2v_\perp u_{n\perp}\cos\varphi'}, \tag{4.25}$$

$$\langle C_{in}[f_i, f_n]\rangle = -R_{in}\left(n_n f_i - n_i\langle f_n\rangle\right), \tag{4.26}$$

$$C_{ni}[f_n, f_i] = -R_{in}\left(n_i f_n - n_n f_i\right), \tag{4.27}$$

$$\langle C_{i,\mathrm{ion}}[f_e, f_n]\rangle = \langle f_n\rangle n_e R_{\mathrm{ion}} \tag{4.28}$$

and

$$C_{n,\mathrm{ion}}[f_e, f_n] = -f_n n_e R_{\mathrm{ion}}. \tag{4.29}$$

The potential $\phi(r,z,t)$ is determined by the quasineutrality equation

$$n_i = n_e. \tag{4.30}$$

To solve this equation, we need to treat the equations implicitly as the potential enters only via its effect on $\partial f_i/\partial t$ and $\partial f_e/\partial t$. The need to use implicit methods is one of the reasons why we are trying to extract some of the low order moments from the distribution function.

5. Wall boundary conditions

We impose wall boundary conditions at $z = 0$ and $z = L_z$. In principle, we need to consider the effect of the magnetic presheath (Chodura 1982) because the magnetic field is not perpendicular to the wall. However, the complicated boundary conditions that the magnetic presheath imposes on drift kinetic models are an active area of research (Geraldini *et al.* 2017, 2018, 2019; Geraldini 2021). To avoid this complication, we assume that the electron gyroradius is much smaller than the Debye length, thus ensuring that electrons are magnetized even within the thin sheath of non-neutral plasma with a width of the order of the Debye length that forms on walls to ensure quasineutrality. With this assumption, we can impose boundary conditions similar to those proposed by Parker *et al.* (1993).

The boundary conditions that we propose make use of the fact that the potential drop across the magnetic presheath and the Debye sheath repels electrons away from the wall because otherwise electrons would flow to the wall at much greater rate than ions due to their lower mass and higher thermal speed. In our model, $\phi(r,0,t)$ and $\phi(r,L_z,t)$ are not the potential of the wall, but the potential at the entrance of the magnetic presheath. In this report, we choose the potential of the wall at $z = 0$ to be 0 without loss of generality. We denote the potential of the wall at $z = L_z$ as ϕ_w. Then, for the magnetic presheaths and Debye sheaths to repel electrons, $\phi(r,0,t)$ must be larger than 0 and $\phi(r,L_z,t)$ must be larger than ϕ_w.

The value of the potential at $z = 0$ and $z = L_z$ is determined by the relationship between the current crossing the magnetic presheath and the Debye sheath and the total potential drop across these layers. We consider the magnetic presheath and the Debye sheath at $z = L_z$ first, and we will then apply the results that we obtain to the magnetic presheath and the Debye sheath at $z = 0$. Ions recombine when they hit the wall, so no ions come back. The velocity of the ions perpendicular to the wall is a combination of parallel velocity and $\mathbf{E}\times\mathbf{B}$ drift, $v_\parallel B_z/B + B^{-1}(\partial\phi/\partial r)$. Thus, the ions that would come back from the wall must satisfy $v_\parallel < -B_z^{-1}(\partial\phi/\partial r)$, giving

$$f_i(r, L_z, v_\parallel < -B_z^{-1}(\partial\phi/\partial r), v_\perp, t) = 0. \tag{5.1}$$

Since the electrons are magnetized, the potential drop across the magnetic presheath and the Debye sheath only modifies the parallel velocity of electrons. Within these layers, the parallel energy $\mathcal{E}_\parallel := m_e v_\parallel^2/2 - e\phi$ is conserved, and as a result an electron that has velocity $v_\parallel$ at the entrance of the sheath is slowed down to a parallel velocity $[v_\parallel^2 - 2e(\phi(r,L_z,t) - \phi_w)/m_e]^{1/2}$ when it reaches the wall. Thus, electrons with a parallel velocity larger than $\sqrt{2e(\phi(r,L_z,t)-\phi_w)/m_e}$ reach the wall, where they recombine with ions, whereas electrons with parallel velocity smaller than $\sqrt{2e(\phi(r,L_z,t)-\phi_w)/m_e}$ are repelled back into the quasineutral plasma. Thus, the boundary condition on the electron distribution function at $z = L_z$ is

$$f_e(r, L_z, v_\| < 0, v_\perp, t) = \begin{cases} f_e(r, L_z, -v_\|, v_\perp, t) & \text{for } v_\| \geqslant -\sqrt{2e(\phi(r, L_z, t) - \phi_w)/m_e}, \\ 0 & \text{for } v_\| < -\sqrt{2e(\phi(r, L_z, t) - \phi_w)/m_e}. \end{cases} \tag{5.2}$$

Here we can neglect the $\mathbf{E} \times \mathbf{B}$ drift because it is small compared to the typical electron thermal speed by a factor of $\sqrt{m_e/m_i} \ll 1$. Expressions (5.1) and (5.2) give the parallel ion and electron current density towards the wall at the entrance of the sheath at $z = L_z$,

$$J_{i\|}(r, L_z, t) = 2\pi e \int_{-B_z^{-1}(\partial\phi/\partial r)}^{\infty} \mathrm{d}v_\| \int_0^\infty \mathrm{d}v_\perp \, v_\perp v_\| f_i(r, L_z, v_\|, v_\perp, t) \tag{5.3}$$

and

$$J_{e\|}(r, L_z, t) = -2\pi e \int_{\sqrt{2e(\phi(r, L_z, t) - \phi_w)/m_e}}^{\infty} \mathrm{d}v_\| \int_0^\infty \mathrm{d}v_\perp \, v_\perp v_\| f_e(r, L_z, v_\|, v_\perp, t). \tag{5.4}$$

Hence, the total parallel current density at $z = L_z$ is the following function of $\phi(r, L_z, t) - \phi_w$ and the radial derivative of $\phi(r, L_z, t) - \phi_w$,

$$\begin{aligned} J_\|(r, L_z, t) = {} & 2\pi e \int_{-B_z^{-1}(\partial\phi/\partial r)}^{\infty} \mathrm{d}v_\| \int_0^\infty \mathrm{d}v_\perp \, v_\perp v_\| f_i(r, L_z, v_\|, v_\perp, t) \\ & -2\pi e \int_{\sqrt{2e(\phi(r, L_z, t) - \phi_w)/m_e}}^{\infty} \mathrm{d}v_\| \int_0^\infty \mathrm{d}v_\perp \, v_\perp v_\| f_e(r, L_z, v_\|, v_\perp, t). \end{aligned} \tag{5.5}$$

We assume that the potential ϕ_w does not depend on r and hence the radial derivative of $\phi(r, L_z, t) - \phi_w$ is simply the radial derivative of $\phi(r, L_z, t)$.

The conditions at $z = 0$ for the ion and electron distribution functions and the potential are similar to those for $z = L_z$. For the ion and electron distribution functions, we find

$$f_i(r, 0, v_\| > -B_z^{-1}(\partial\phi/\partial r), v_\perp, t) = 0 \tag{5.6}$$

and

$$f_e(r, 0, v_\| > 0, v_\perp, t) = \begin{cases} f_e(r, 0, -v_\|, v_\perp, t) & \text{for } v_\| \leqslant \sqrt{2e\phi(r, 0, t)/m_e}, \\ 0 & \text{for } v_\| > \sqrt{2e\phi(r, 0, t)/m_e}, \end{cases} \tag{5.7}$$

The relationship between the parallel current and the potential at the magnetic presheath entrance is

$$\begin{aligned} J_\|(r, 0, t) = {} & -2\pi e \int_{-\infty}^{-B_z^{-1}(\partial\phi/\partial r)} \mathrm{d}v_\| \int_0^\infty \mathrm{d}v_\perp \, v_\perp v_\| f_i(r, 0, v_\|, v_\perp, t) \\ & +2\pi e \int_{-\infty}^{-\sqrt{2e\phi(r, 0, t)/m_e}} \mathrm{d}v_\| \int_0^\infty \mathrm{d}v_\perp \, v_\perp v_\| f_e(r, 0, v_\|, v_\perp, t). \end{aligned} \tag{5.8}$$

We still need boundary conditions for the neutral distribution function. The neutrals hit the wall and thermalize at the temperature of the wall T_w, while also receiving back the ions that have recombined at the wall, that is,

$$f_n(r, 0, v_r, v_z > 0, v_\zeta, t) = \Gamma_0 f_{Kw}\left(v_z, \sqrt{v_r^2 + v_\zeta^2}\right) \tag{5.9}$$

and

$$f_n(r, L_z, v_r, v_z < 0, v_\zeta, t) = \Gamma_L f_{Kw}\left(v_z, \sqrt{v_r^2 + v_\zeta^2}\right), \tag{5.10}$$

where

$$f_{Kw}(v_n, v_t) := \frac{3}{4\pi}\left(\frac{m_i}{T_w}\right)^2 \frac{|v_n|}{\sqrt{v_n^2+v_t^2}} \exp\left(-\frac{m_i(v_n^2+v_t^2)}{2T_w}\right) \tag{5.11}$$

is the Knudsen cosine distribution (Knudsen 1916), and

$$\Gamma_0 := 2\pi \int_{-\infty}^{-B_z^{-1}(\partial\phi/\partial r)} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp\, v_\perp \left|\frac{v_\parallel B_z}{B} + \frac{1}{B}\frac{\partial\phi}{\partial r}\right| f_i(r, 0, v_\parallel, v_\perp, t) + \int_{-\infty}^{\infty} \mathrm{d}v_r \int_{-\infty}^{0} \mathrm{d}v_z \int_{-\infty}^{\infty} \mathrm{d}v_\zeta\, |v_z| f_n(r, 0, v_r, v_z, v_\zeta, t) \tag{5.12}$$

and

$$\Gamma_L := 2\pi \int_{-B_z^{-1}(\partial\phi/\partial r)}^{\infty} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp\, v_\perp \left(\frac{v_\parallel B_z}{B} + \frac{1}{B}\frac{\partial\phi}{\partial r}\right) f_i(r, L_z, v_\parallel, v_\perp, t) + \int_{-\infty}^{\infty} \mathrm{d}v_r \int_{0}^{\infty} \mathrm{d}v_z \int_{-\infty}^{\infty} \mathrm{d}v_\zeta\, v_z f_n(r, L_z, v_r, v_z, v_\zeta, t) \tag{5.13}$$

are the fluxes of neutrals and ions towards the walls at $z = 0$ and $z = L_z$.

6. 2D moment drift kinetics

Instead of solving for $f_s(r, z, v_\parallel, v_\perp, t)$ with $s = i, e$, we solve for

$$F_s(r, z, w_\parallel, w_\perp, t) := \frac{v_{ts}^3(r, z, t)}{n_s(r, z, t)} f_s\Big(r, z, u_{s\parallel}(r, z, t) + v_{ts}(r, z, t) w_\parallel, v_{ts}(r, z, t) w_\perp, t\Big), \tag{6.1}$$

where we have defined the normalized velocities

$$w_\parallel(r, z, v_\parallel, t) := \frac{v_\parallel - u_{s\parallel}(r, z, t)}{v_{ts}(r, z, t)} \tag{6.2}$$

and

$$w_\perp(r, z, v_\perp, t) := \frac{v_\perp}{v_{ts}(r, z, t)}, \tag{6.3}$$

the average parallel velocity

$$u_{s\parallel}(r, z, t) := \frac{2\pi}{n_s} \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp\, v_\perp v_\parallel f_s(r, z, v_\parallel, v_\perp, t) \tag{6.4}$$

and the thermal speed

$$v_{ts}(r, z, t) := \sqrt{\frac{2T_s(r, z, t)}{m_s}}, \tag{6.5}$$

with

$$T_s(r, z, t) := \frac{2\pi}{n_s} \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp\, v_\perp \frac{m_s[(v_\parallel - u_{s\parallel}(r, z, t))^2 + v_\perp^2]}{3} f_s(r, z, v_\parallel, v_\perp, t) \tag{6.6}$$

the temperature of species s. According to its definition, $F_s(r, z, w_\parallel, w_\perp, t)$ must satisfy the conditions

$$2\pi \int_{-\infty}^{\infty} \mathrm{d}w_\parallel \int_0^\infty \mathrm{d}w_\perp\, w_\perp F_s(r, z, w_\parallel, w_\perp, t) = 1, \tag{6.7}$$

$$2\pi \int_{-\infty}^{\infty} \mathrm{d}w_\parallel \int_0^\infty \mathrm{d}w_\perp\, w_\perp w_\parallel F_s(r, z, w_\parallel, w_\perp, t) = 0 \tag{6.8}$$

and

$$2\pi \int_{-\infty}^{\infty} \mathrm{d}w_\parallel \int_0^\infty \mathrm{d}w_\perp \, w_\perp (w_\parallel^2 + w_\perp^2) F_s(r, z, w_\parallel, w_\perp, t) = \frac{3}{2} \tag{6.9}$$

at every point (r, z) and time t.

Similarly, for neutrals, we solve for

$$F_n(r, z, \underbrace{w_r, w_z, w_\zeta}_{=\mathbf{w}}, t) := \frac{v_{tn}^3(r, z, t)}{n_n(r, z, t)} f_n\Big(r, z, \mathbf{u}_n(r, z, t) + v_{tn}(r, z, t)\mathbf{w}, t\Big), \tag{6.10}$$

where we have defined the neutral temperature

$$T_n(r, z, t) := \frac{1}{n_n} \int \frac{m_i |\mathbf{v} - \mathbf{u}_n(r, z, t)|^2}{3} f_n(r, z, \mathbf{v}, t) \, \mathrm{d}^3 v. \tag{6.11}$$

According to its definition, $F_n(r, z, \mathbf{w}, t)$ must satisfy the conditions

$$\int F_n(r, z, \mathbf{w}, t) \, \mathrm{d}^3 w = 1, \tag{6.12}$$

$$\int \mathbf{w} F_n(r, z, \mathbf{w}, t) \, \mathrm{d}^3 w = 0 \tag{6.13}$$

and

$$\int w^2 F_n(r, z, \mathbf{w}, t) \, \mathrm{d}^3 w = \frac{3}{2} \tag{6.14}$$

at every point (r, z) and time t.

6.1. *Ion equations*

The equations for n_i, $u_{i\parallel}$ and T_i are

$$\frac{\partial n_i}{\partial t} - \frac{\partial}{\partial r}\left(\frac{n_i}{B}\frac{\partial \phi}{\partial z}\right) + \frac{\partial}{\partial z}\left[n_i\left(\frac{u_{i\parallel} B_z}{B} + \frac{1}{B}\frac{\partial \phi}{\partial r}\right)\right] = n_n n_e R_{\mathrm{ion}} + \int S_i \, \mathrm{d}^3 v, \tag{6.15}$$

$$n_i m_i \left[\frac{\partial u_{i\parallel}}{\partial t} - \frac{1}{B}\frac{\partial \phi}{\partial z}\frac{\partial u_{i\parallel}}{\partial r} + \left(\frac{u_{i\parallel} B_z}{B} + \frac{1}{B}\frac{\partial \phi}{\partial r}\right)\frac{\partial u_{i\parallel}}{\partial z}\right] = -\frac{B_z}{B}\frac{\partial p_{i\parallel}}{\partial z} - \frac{e n_i B_z}{B}\frac{\partial \phi}{\partial z}$$
$$+ n_i m_i (n_n R_{in} + n_e R_{\mathrm{ion}})(u_{n\parallel} - u_{i\parallel}) + \int m_i (v_\parallel - u_{i\parallel}) S_i \, \mathrm{d}^3 v \tag{6.16}$$

and

$$\frac{3}{2} n_i \left[\frac{\partial T_i}{\partial t} - \frac{1}{B}\frac{\partial \phi}{\partial z}\frac{\partial T_i}{\partial r} + \left(\frac{u_{i\parallel} B_z}{B} + \frac{1}{B}\frac{\partial \phi}{\partial r}\right)\frac{\partial T_i}{\partial z}\right] = -\frac{B_z}{B}\frac{\partial q_{i\parallel}}{\partial z} - \frac{p_{i\parallel} B_z}{B}\frac{\partial u_{i\parallel}}{\partial z}$$
$$+ \frac{3}{2} n_i (n_n R_{in} + n_e R_{\mathrm{ion}})(T_n - T_i) + \frac{1}{2} n_i m_i (n_n R_{in} + n_e R_{\mathrm{ion}})[(u_{n\parallel} - u_{i\parallel})^2 + u_{n\perp}^2]$$
$$+ \frac{3 n_e m_e \nu_{ei}}{m_i}(T_e - T_i) + \int \left(\frac{1}{2} m_i |\mathbf{v} - u_{i\parallel}\hat{\mathbf{z}}|^2 - \frac{3}{2} T_i\right) S_i \, \mathrm{d}^3 v. \tag{6.17}$$

Here,

$$\nu_{ei} := \frac{16\sqrt{\pi}}{3} \frac{e^4 n_i \ln \Lambda}{(4\pi\epsilon_0)^2 m_e^2 v_{te}^3} \tag{6.18}$$

is the electron-ion collision frequency as defined by Braginskii (Braginskii 1958), and we have defined the parallel pressure

$$p_{s\parallel}[F_s, n_s, v_{ts}](r, z, t) := 2\pi n_s m_s v_{ts}^2 \int_{-\infty}^{\infty} \mathrm{d}w_\parallel \int_0^\infty \mathrm{d}w_\perp \, w_\perp w_\parallel^2 F_s(r, z, w_\parallel, w_\perp, t) \tag{6.19}$$

and the parallel heat flux

$$q_{s\parallel}[F_s, n_s, v_{ts}](r,z,t) := \pi n_s m_s v_{ts}^3 \int_{-\infty}^{\infty} \mathrm{d}w_\parallel \int_0^\infty \mathrm{d}w_\perp\, w_\perp w_\parallel (w_\parallel^2 + w_\perp^2) F_s(r,z,w_\parallel,w_\perp,t) \tag{6.20}$$

for the charged species $s = i, e$.

The ion kinetic equation is

$$\frac{\partial F_i}{\partial t} + \dot{r}_i \frac{\partial F_i}{\partial r} + \dot{z}_i \frac{\partial F_i}{\partial z} + \dot{w}_{\parallel i} \frac{\partial F_i}{\partial w_\parallel} + \dot{w}_{\perp i} \frac{\partial F_i}{\partial w_\perp} = \dot{F}_i + \mathcal{C}_{ii} + \mathcal{C}_{in} + \mathcal{C}_{i,\mathrm{ion}} + \mathcal{S}_i. \tag{6.21}$$

Here, we have defined the coefficients

$$\dot{r}_i[\phi](r,z,t) := -\frac{1}{B}\frac{\partial \phi}{\partial z}, \tag{6.22}$$

$$\dot{z}_i[u_{i\parallel}, v_{ti}, \phi](r,z,w_\parallel,t) := \frac{u_{i\parallel} B_z}{B} + \frac{1}{B}\frac{\partial \phi}{\partial r} + \frac{v_{ti} B_z}{B} w_\parallel, \tag{6.23}$$

$$\begin{aligned} \dot{w}_{\parallel i}[F_i, n_i, u_{i\parallel}, v_{ti}](r,z,w_\parallel,t) := {} & \frac{B_z}{n_i m_i v_{ti} B}\frac{\partial p_{i\parallel}}{\partial z} \\ & + \frac{2 w_\parallel B_z}{3 n_i m_i v_{ti}^2 B}\left[\frac{\partial q_{i\parallel}}{\partial z} + \left(p_{i\parallel} - \frac{3}{2} n_i m_i v_{ti}^2\right)\frac{\partial u_{i\parallel}}{\partial z}\right] - \frac{w_\parallel^2 B_z}{B}\frac{\partial v_{ti}}{\partial z}, \end{aligned} \tag{6.24}$$

$$\dot{w}_{\perp i}[F_i, n_i, u_{i\parallel}, v_{ti}](r,z,w_\parallel,w_\perp,t) := \frac{2 w_\perp B_z}{3 n_i m_i v_{ti}^2 B}\left(\frac{\partial q_{i\parallel}}{\partial z} + p_{i\parallel}\frac{\partial u_{i\parallel}}{\partial z}\right) - \frac{w_\parallel w_\perp B_z}{B}\frac{\partial v_{ti}}{\partial z} \tag{6.25}$$

and

$$\begin{aligned} \dot{F}_i[F_i, n_i, u_{i\parallel}, v_{ti}](r,z,w_\parallel,w_\perp,t) := {} & \frac{B_z}{B}\left[w_\parallel\left(3\frac{\partial v_{ti}}{\partial z} - \frac{v_{ti}}{n_i}\frac{\partial n_i}{\partial z}\right)\right. \\ & \left. - \frac{2}{n_i m_i v_{ti}^2}\left(\frac{\partial q_{i\parallel}}{\partial z} + \left(p_{i\parallel} - \frac{1}{2} n_i m_i v_{ti}^2\right)\frac{\partial u_{i\parallel}}{\partial z}\right)\right] F_i. \end{aligned} \tag{6.26}$$

We have also defined a modified source $\mathcal{S}_i$ and several modified collision operators. The modified source for the charged species $s = i, e$ is given by

$$\begin{aligned} & \mathcal{S}_s[S_s, F_s, n_s, u_{s\parallel}, v_{ts}](r,z,w_\parallel,w_\perp,t) \\ & := -\left[\frac{F_s}{n_s}\int S_s\,\mathrm{d}^3v - \frac{v_{ts}^3}{n_s} S_s(r,z,u_{s\parallel} + v_{ts} w_\parallel, v_{ts} w_\perp, t)\right] \\ & \quad + \frac{\partial}{\partial w_\parallel}\left[F_s\left(\frac{1}{n_s v_{ts}}\int (v_\parallel - u_{s\parallel}) S_s\,\mathrm{d}^3v + \frac{w_\parallel}{3 n_s v_{ts}^2}\int\left(|\mathbf{v} - u_{s\parallel}\hat{\mathbf{z}}|^2 - \frac{3}{2}v_{ts}^2\right) S_s\,\mathrm{d}^3v\right)\right] \\ & \quad + \frac{1}{w_\perp}\frac{\partial}{\partial w_\perp}\left[\frac{w_\perp^2 F_s}{3 n_s v_{ts}^2}\int\left(|\mathbf{v} - u_{s\parallel}\hat{\mathbf{z}}|^2 - \frac{3}{2}v_{ts}^2\right) S_s\,\mathrm{d}^3v\right]. \end{aligned} \tag{6.27}$$

Note that the differential terms in this modified source could have been included in the definitions of the coefficients $\dot{w}_{\parallel i}$, $\dot{w}_{\perp i}$ and $\dot{F}_i$, but we have decided to make them part of a modified source instead to separate the effect of the source clearly. We will do the same for collisions. This split should not be taken as a suggestion on how to implement these terms in a code. The modified collisions operators are described in Appendix A.

6.2. *Electron equations*

For the electrons, we use the expansion in m_e/m_i described in report 2047357-TN-05-01 M1.3. We first describe the electron fluid equations and how to use them.

- The electron continuity equation is

$$\frac{\partial n_e}{\partial t} - \frac{\partial}{\partial r}\left(\frac{n_e}{B}\frac{\partial \phi}{\partial z}\right) + \frac{\partial}{\partial z}\left[n_e\left(\frac{u_{e\|}B_z}{B} + \frac{1}{B}\frac{\partial \phi}{\partial r}\right)\right] = -n_e n_n R_{\text{ion}} + \int S_e \,\mathrm{d}^3 v. \tag{6.28}$$

Subtracting this equation from equation (6.15) and using quasineutrality, we obtain the current conservation equation

$$\frac{B_z}{B}\frac{\partial}{\partial z}\left[n_e\left(u_{i\|} - u_{e\|}\right)\right] = 0, \tag{6.29}$$

where we have used property (4.7). This equation can be used to calculate $u_{e\|}$ once $u_{e\|}$ is known at $z = 0$. We discuss how to obtain $u_{e\|}$ at $z = 0$ in the next bullet point.

- The electron parallel momentum equation simplifies to

$$0 = -\frac{B_z}{B}\frac{\partial p_{e\|}}{\partial z} + \frac{e n_e B_z}{B}\frac{\partial \phi}{\partial z} + F_{ei\|} + n_e m_e n_n R_{en}(u_{n\|} - u_{e\|}), \tag{6.30}$$

where

$$\begin{aligned} &F_{ei\|}[F_e, n_e, n_i, u_{e\|}, u_{e\|}, v_{te}, v_{ti}](z,t) \\ &\quad := -\frac{8\pi^2 e^4 n_e n_i \ln\Lambda}{(4\pi\epsilon_0)^2 m_e v_{te}^2}\int_{-\infty}^{\infty}\mathrm{d}w_\| \int_0^\infty \mathrm{d}w_\perp \frac{w_\perp (w_\| - (u_{i\|} - u_{e\|})/v_{te}) F_e}{[(w_\| - (u_{i\|} - u_{e\|})/v_{te})^2 + w_\perp^2]^{3/2}} \end{aligned} \tag{6.31}$$

is the friction force between electrons and ions. Equation (6.30) can be used to calculate the potential difference between $z = 0$ and any value of z, $\phi(r,z,t) - \phi(r,0,t)$. To completely determine the potential, we need to calculate $\phi(r,0,t)$. We do so with the current-potential relationships of the magnetic presheath and the Debye sheath, given in equations (5.5) and (5.8). We use two coupled nonlinear equations for the unknowns $\phi(r,0,t)$ and $\phi(r,L_z,t)$.

  - ∘ Integrating equation (6.29) between $z = 0$ and $z = L_z$, we obtain the condition $J_\|(r,L_z,t) - J_\|(r,0,t) = 0$. Since equations (5.5) and (5.8) give $J_\|(r,L_z,t)$ and $J_\|(r,0,t)$ as functions of $\phi(r,L_z,t) - \phi_w$ and $\phi(r,0,t)$, condition $J_\|(r,L_z,t) - J_\|(r,0,t) = 0$ is an equation for $\phi(r,L_z,t)$ and $\phi(r,0,t)$ (we assume the bias ϕ_w to be externally determined).
  - ∘ The other equation is the value of $\phi(r,L_z,t) - \phi(r,0,t)$ obtained by integrating equation (6.30) from $z = 0$ to $z = L_z$. Note that the value of $\phi(r,L_z,t) - \phi(r,0,t)$ depends on the unknown $u_{e\|}(r,0,t)$ (recall that $u_{e\|}$ can be determined everywhere from equation (6.29) for a given $u_{e\|}(r,0,t)$). The value $u_{e\|}(r,0,t)$ depends on $\phi(r,0,t)$ via equation (5.8) and so, in the end, equation (6.30) gives a relationship between $\phi(r,L_z,t) - \phi(r,0,t)$ and $\phi(r,0,t)$.

Once these two equations for $\phi(r,0,t)$ and $\phi(r,L_z,t)$ are solved, we can substitute the value of $\phi(r,0,t)$ in equation (5.8) to calculate $u_{e\|}(r,0,t)$, and then integrate equation (6.29) to find $u_{e\|}$ everywhere.

- The electron energy equation is

$$\frac{3}{2}n_e\left[\frac{\partial T_e}{\partial t}-\frac{1}{B}\frac{\partial\phi}{\partial z}\frac{\partial T_e}{\partial r}+\left(\frac{u_{e\|}B_z}{B}+\frac{1}{B}\frac{\partial\phi}{\partial r}\right)\frac{\partial T_e}{\partial z}\right]=-\frac{B_z}{B}\frac{\partial q_{e\|}}{\partial z}-\frac{p_{e\|}B_z}{B}\frac{\partial u_{e\|}}{\partial z}$$
$$-n_e n_n R_{\text{ion}}E_{\text{ion}}+\frac{3n_e m_e \nu_{ei}}{m_i}(T_i-T_e)+F_{ei\|}(u_{i\|}-u_{e\|})$$
$$+\frac{3n_e m_e n_n R_{en}}{m_i}(T_n-T_e)+n_e m_e n_n R_{en}[(u_{n\|}-u_{e\|})^2+u_{n\perp}^2]$$
$$+\int\left(\frac{1}{2}m_e|\mathbf{v}-u_{e\|}\hat{\mathbf{z}}|^2-\frac{3}{2}T_e\right)S_e\,\mathrm{d}^3v, \quad (6.32)$$

where E_{ion} is the ionization energy cost that includes in it radiation from excited states.

The kinetic equation for electrons is

$$\dot{z}_e\frac{\partial F_e}{\partial z}+\dot{w}_{\|e}\frac{\partial F_e}{\partial w_\|}+\dot{w}_{\perp e}\frac{\partial F_e}{\partial w_\perp}=\dot{F}_e+\mathcal{C}_{ee}+\mathcal{C}_{ei}+\mathcal{C}_{en}, \quad (6.33)$$

where

$$\dot{z}_e[F_e,v_{te}](z,w_\|,t):=v_{te}w_\|, \quad (6.34)$$

$$\dot{w}_{\|e}[F_e,u_{e\|},v_{te}](z,w_\|,t):=\frac{B_z}{n_e m_e v_{te}B}\frac{\partial p_{e\|}}{\partial z}+\frac{2w_\|B_z}{3n_e m_e v_{te}^2 B}\frac{\partial q_{e\|}}{\partial z}-\frac{w_\|^2 B_z}{B}\frac{\partial v_{te}}{\partial z}, \quad (6.35)$$

$$\dot{w}_{\perp e}[F_e,u_{e\|},v_{te}](z,w_\|,w_\perp,t):=\frac{2w_\perp B_z}{3n_e m_e v_{te}^2 B}\frac{\partial q_{e\|}}{\partial z}-\frac{w_\|w_\perp B_z}{B}\frac{\partial v_{te}}{\partial z} \quad (6.36)$$

and

$$\dot{F}_e[F_e,u_{e\|},v_{te}](z,w_\|,w_\perp,t):=\frac{B_z}{B}\left[w_\|\left(3\frac{\partial v_{te}}{\partial z}-\frac{v_{te}}{n_e}\frac{\partial n_e}{\partial z}\right)-\frac{2}{n_e m_e v_{te}^2}\frac{\partial q_{e\|}}{\partial z}\right]F_e. \quad (6.37)$$

The modified collision operators $\mathcal{C}_{ee}$, $\mathcal{C}_{ei}$ and $\mathcal{C}_{en}$ are described in Appendix B.

6.3. *Neutral equations*

The fluid equations for the neutrals are

$$\frac{\partial n_n}{\partial t}+\nabla\cdot(n_n\mathbf{u}_n)=-n_n n_e R_{\text{ion}}+\int S_n\,\mathrm{d}^3v, \quad (6.38)$$

$$n_n m_i\left(\frac{\partial\mathbf{u}_n}{\partial t}+\mathbf{u}_n\cdot\nabla\mathbf{u}_n\right)=-\nabla\cdot\mathbf{P}_n+n_n m_i n_i R_{in}\left(u_{i\|}\hat{\boldsymbol{\zeta}}-\mathbf{u}_n\right)$$
$$+\int m_i(\mathbf{v}-\mathbf{u}_n)S_n\,\mathrm{d}^3v \quad (6.39)$$

and

$$\frac{3}{2}n_n\left(\frac{\partial T_n}{\partial t}+\mathbf{u}_n\cdot\nabla T_n\right)=-\nabla\cdot\mathbf{q}_n-\mathbf{P}_n:\nabla\mathbf{u}_n+\frac{3}{2}n_n n_i R_{in}(T_i-T_n)$$
$$+\frac{1}{2}n_n m_i n_i R_{in}[(u_{n\|}-u_{i\|})^2+u_{n\perp}^2]+\frac{3n_e m_e n_n R_{en}}{m_i}(T_e-T_n)$$
$$+\int\left(\frac{1}{2}m_i|\mathbf{v}-\mathbf{u}_n|^2-\frac{3}{2}T_n\right)S_n\,\mathrm{d}^3v. \quad (6.40)$$

Here, we have defined the pressure tensor

$$\mathbf{P}_n[F_n, n_n, v_{tn}](r, z, t) := n_n m_i v_{tn}^2 \int \mathbf{w}\mathbf{w} F_n(r, z, \mathbf{w}, t)\, \mathrm{d}^3 w \tag{6.41}$$

and the heat flux

$$\mathbf{q}_n[F_n, n_n, v_{tn}](z, t) := \frac{1}{2} n_n m_i v_{tn}^3 \int w^2 \mathbf{w} F_n(r, z, \mathbf{w}, t)\, \mathrm{d}^3 w. \tag{6.42}$$

The neutral kinetic equation is

$$\frac{\partial F_n}{\partial t} + \dot{r}_n \frac{\partial F_n}{\partial r} + \dot{z}_n \frac{\partial F_n}{\partial z} + \dot{\mathbf{w}}_n \cdot \nabla_w F_n = \dot{F}_n + \mathcal{C}_{ni} + \mathcal{S}_n, \tag{6.43}$$

where we have defined the coefficients

$$\dot{r}_n[u_{nr}, v_{tn}](r, z, w_r, t) := u_{nr} + v_{tn} w_r, \tag{6.44}$$

$$\dot{z}_i[u_{nz}, v_{tn}](r, z, w_z, t) := u_{nz} + v_{tn} w_z, \tag{6.45}$$

$$\begin{aligned} \dot{\mathbf{w}}_n[F_n, n_n, \mathbf{u}_n, v_{tn}](r, z, \mathbf{w}, t) &:= \frac{1}{n_n m_i v_{tn}} \nabla \cdot \mathbf{P}_n \\ &+ \frac{2\mathbf{w}}{3 n_n m_i v_{tn}^2} \left(\nabla \cdot \mathbf{q}_n + \mathbf{P}_n : \nabla \mathbf{u}_n \right) - \mathbf{w} \cdot \nabla \mathbf{u}_n - \mathbf{w}\mathbf{w} \cdot \nabla v_{tn}, \end{aligned} \tag{6.46}$$

The modified charge exchange collision operator $\mathcal{C}_{ni}$ is described in Appendix C. The modified source is

$$\begin{aligned} \mathcal{S}_n[S_n, F_n, n_n, \mathbf{u}_n, v_{tn}](r, z, \mathbf{w}, t) := -\left[\frac{F_n}{n_n} \int S_n \, \mathrm{d}^3 v - \frac{v_{tn}^3}{n_n} S_s(r, z, \mathbf{u}_n + v_{tn}\mathbf{w}, t) \right] \\ + \nabla_w \cdot \left[F_n \left(\frac{1}{n_n v_{tn}} \int (\mathbf{v} - \mathbf{u}_n) S_n \, \mathrm{d}^3 v + \frac{\mathbf{w}}{3 n_n v_{tn}^2} \int \left(|\mathbf{v} - \mathbf{u}_n|^2 - \frac{3}{2} v_{tn}^2 \right) S_n \, \mathrm{d}^3 v \right) \right]. \end{aligned} \tag{6.47}$$

6.4. *Boundary conditions*

These equations have to be solved with the boundary conditions in equations (5.1), (5.2), (5.6), (5.7), (5.9) and (5.10). For n_s, $u_{s\|}$, v_{ts} and F_s known at time t, we can construct f_s at $z = 0$ and $z = L_z$, and we can apply the wall boundary conditions to f_s. We can then use the resulting f_s to obtain n_s, $\mathbf{u}_s$, v_{ts} and F_s, closing the system of equations.

7. Discussion

The model that we propose is comprised of:

- the three fluid equations (6.15), (6.16) and (6.17) for ions that have to be solved in conjunction with the ion kinetic equation (6.21);
- the five fluid equations (6.38), (6.39) and (6.40) for neutrals that have to be solved in conjunction with the neutral kinetic equation (6.43);
- the two fluid equations (6.29) and (6.32) for electrons that have to be solved in conjunction with the electron kinetic equation (6.33); and
- the electron parallel momentum equation (6.30) for the potential.

The boundary conditions for this system of equations are described in section 5.

To test the model proposed in this report, we will first extend the existing 1D code based on adiabatic electrons to wall boundary conditions. We will then explore the effect of adding electrons. For most physics of interest, it is sufficient to use simplified ion-ion and electron-electron collision operators, and for this reason we do not expect to implement a full Fokker-Planck collision operator.

Appendix A. Modified collision operators for the ion kinetic equation

The modified Fokker-Planck like-particle collision operator is

$$\begin{aligned}\mathcal{C}_{ss}[F_s,n_s,v_{ts}](r,z,w_\parallel,w_\perp,t)\\ :=&\frac{2\pi e^4 n_s\ln\Lambda}{(4\pi\epsilon_0)^2 m_s^2 v_{ts}^3}\left\{\frac{\partial}{\partial w_\parallel}\left(\mathcal{D}_{\parallel\parallel}[F_s]\frac{\partial F_s}{\partial w_\parallel}+\mathcal{D}_{\parallel\perp}[F_s]\frac{\partial F_s}{\partial w_\perp}+\mathcal{P}_\parallel[F_s]F_s\right)\right.\\ &\left.+\frac{1}{w_\perp}\frac{\partial}{\partial w_\perp}\left[w_\perp\left(\mathcal{D}_{\parallel\perp}[F_s]\frac{\partial F_s}{\partial w_\parallel}+\mathcal{D}_{\perp\perp}[F_s]\frac{\partial F_s}{\partial w_\perp}+\mathcal{P}_\perp[F_s]F_s\right)\right]\right\}.\end{aligned} \tag{A 1}$$

The coefficients needed for this collision operator are

$$\begin{aligned}\mathcal{D}_{\parallel\parallel}[F_s](r,z,w_\parallel,w_\perp,t):=4\int_{-\infty}^{\infty}\mathrm{d}w_\parallel'\int_0^\infty\mathrm{d}w_\perp'\,\frac{w_\perp'}{\sqrt{(w_\parallel-w_\parallel')^2+(w_\perp+w_\perp')^2}}\\ \times\left(K(\kappa)-\frac{(w_\parallel-w_\parallel')^2E(\kappa)}{(w_\parallel-w_\parallel')^2+(w_\perp-w_\perp')^2}\right)F_s(z,w_\parallel',w_\perp',t),\end{aligned} \tag{A 2}$$

$$\begin{aligned}\mathcal{D}_{\parallel\perp}[F_s](r,z,w_\parallel,w_\perp,t):=2\int_{-\infty}^{\infty}\mathrm{d}w_\parallel'\int_0^\infty\mathrm{d}w_\perp'\,\frac{w_\perp'(w_\parallel-w_\parallel')}{w_\perp\sqrt{(w_\parallel-w_\parallel')^2+(w_\perp+w_\perp')^2}}\\ \times\left(\frac{[(w_\parallel-w_\parallel')^2-w_\perp^2+w_\perp'^2]E(\kappa)}{(w_\parallel-w_\parallel')^2+(w_\perp-w_\perp')^2}-K(\kappa)\right)F_s(z,w_\parallel',w_\perp',t),\end{aligned} \tag{A 3}$$

$$\begin{aligned}\mathcal{D}_{\perp\perp}[F_s](r,z,w_\parallel,w_\perp,t):=2\int_{-\infty}^{\infty}\mathrm{d}w_\parallel'\int_0^\infty\mathrm{d}w_\perp'\,\frac{w_\perp'}{w_\perp^2\sqrt{(w_\parallel-w_\parallel')^2+(w_\perp+w_\perp')^2}}\\ \times\left\{2w_\perp\left[\frac{w_\perp(w_\parallel-w_\parallel')^2}{(w_\parallel-w_\parallel')^2+(w_\perp-w_\perp)^2}-w_\perp'\right]E(\kappa)\right.\\ \left.+[(w_\parallel-w_\parallel')^2+w_\perp^2+w_\perp'^2][K(\kappa)-E(\kappa)]\right\}F_s(z,w_\parallel',w_\perp',t),\end{aligned} \tag{A 4}$$

$$\begin{aligned}\mathcal{P}_\parallel[F_s](r,z,w_\parallel,w_\perp,t):=8\int_{-\infty}^{\infty}\mathrm{d}w_\parallel'\int_0^\infty\mathrm{d}w_\perp'\,\frac{w_\perp'(w_\parallel-w_\parallel')}{\sqrt{(w_\parallel-w_\parallel')^2+(w_\perp+w_\perp')^2}}\\ \times\left(\frac{K(\kappa)-E(\kappa)}{(w_\parallel-w_\parallel')^2+(w_\perp+w_\perp')^2}-\frac{E(\kappa)}{(w_\parallel-w_\parallel')^2+(w_\perp-w_\perp')^2}\right)F_s(z,w_\parallel',w_\perp',t)\end{aligned} \tag{A 5}$$

and

$$\mathcal{P}_\perp[F_s](r,z,w_\parallel,w_\perp,t) := 4\int_{-\infty}^{\infty} \mathrm{d}w'_\parallel \int_0^\infty \mathrm{d}w'_\perp \, \frac{w'_\perp}{w_\perp\sqrt{(w_\parallel - w'_\parallel)^2 + (w_\perp + w'_\perp)^2}} \times \left(\frac{[(w_\parallel - w'_\parallel)^2 - w_\perp^2 + w'^2_\perp]E(\kappa)}{(w_\parallel - w'_\parallel)^2 + (w_\perp - w'_\perp)^2} - K(\kappa) \right) F_s(z, w'_\parallel, w'_\perp, t). \quad \text{(A 6)}$$

Here, $K(\kappa) := \int_0^{\pi/2}(1-\kappa^2\sin^2\alpha)^{-1/2}\,\mathrm{d}\alpha$ and $E(\kappa) := \int_0^{\pi/2}(1-\kappa^2\sin^2\alpha)^{1/2}\,\mathrm{d}\alpha$ are the elliptic integrals, and the function κ is

$$\kappa(w_\parallel, w_\perp, w'_\parallel, w'_\perp) := \sqrt{\frac{4w_\perp w'_\perp}{(w_\parallel - w'_\parallel)^2 + (w_\perp + w'_\perp)^2}}. \quad \text{(A 7)}$$

The modified charge exchange collision operator for the ion kinetic equation is

$$\begin{aligned}
&\mathcal{C}_{in}[F_i, \langle F_n\rangle, n_n, u_{i\parallel}, \mathbf{u}_n, v_{ti}, v_{tn}](z, w_\parallel, w_\perp, t) \\
&:= - n_n R_{in}\left[F_i - \frac{v_{ti}^3}{v_{tn}^3}\langle F_n\rangle\left(r, z, \frac{u_{i\parallel} - u_{n\parallel}}{v_{tn}} + \frac{v_{ti}}{v_{tn}}w_\parallel, \frac{v_{ti}}{v_{tn}}w_\perp, t\right)\right] \\
&+ n_n R_{in}\frac{\partial}{\partial w_\parallel}\left[\left(\frac{u_{n\parallel} - u_{i\parallel}}{v_{ti}} + \frac{w_\parallel}{2}\left(\frac{v_{tn}^2}{v_{ti}^2} - 1 + \frac{2[(u_{n\parallel} - u_{i\parallel})^2 + u_{n\perp}^2]}{3v_{ti}^2}\right)\right)F_i\right] \\
&+ \frac{n_n R_{in}}{w_\perp}\frac{\partial}{\partial w_\perp}\left[\frac{w_\perp^2}{2}\left(\frac{v_{tn}^2}{v_{ti}^2} - 1 + \frac{2[(u_{n\parallel} - u_{i\parallel})^2 + u_{n\perp}^2]}{3v_{ti}^2}\right)F_i\right], \qquad \text{(A 8)}
\end{aligned}$$

where

$$\langle F_n\rangle[F_n, \mathbf{u}_{n\perp}](r, z, w_\parallel, w_\perp, t) := \frac{1}{2\pi}\int_0^{2\pi} F_n\left(r, z, w_\perp\cos\varphi - \frac{u_{nr}}{v_{tn}}, w_\perp\sin\varphi - \frac{u_{nz}}{v_{tn}}, w_\parallel, t\right)\mathrm{d}\varphi \quad \text{(A 9)}$$

is the gyroaverage of F_n.

Finally, the modified ionization collision operator for the ion kinetic equation is

$$\begin{aligned}
&\mathcal{C}_{i,\mathrm{ion}}[\langle F_n\rangle, n_e, u_{i\parallel}, \mathbf{u}_n, v_{ti}, v_{tn}](r, z, w_\parallel, w_\perp, t) \\
&:= - n_e R_{\mathrm{ion}}\left[F_i - \frac{v_{ti}^3}{v_{tn}^3}\langle F_n\rangle\left(r, z, \frac{u_{i\parallel} - u_{n\parallel}}{v_{tn}} + \frac{v_{ti}}{v_{tn}}w_\parallel, \frac{v_{ti}}{v_{tn}}w_\perp, t\right)\right] \\
&+ n_e R_{\mathrm{ion}}\frac{\partial}{\partial w_\parallel}\left[\left(\frac{u_{n\parallel} - u_{i\parallel}}{v_{ti}} + \frac{w_\parallel}{2}\left(\frac{v_{tn}^2}{v_{ti}^2} - 1 + \frac{2[(u_{n\parallel} - u_{i\parallel})^2 + u_{n\perp}^2]}{3v_{ti}^2}\right)\right)F_i\right] \\
&+ \frac{n_e R_{\mathrm{ion}}}{w_\perp}\frac{\partial}{\partial w_\perp}\left[\frac{w_\perp^2}{2}\left(\frac{v_{tn}^2}{v_{ti}^2} - 1 + \frac{2[(u_{n\parallel} - u_{i\parallel})^2 + u_{n\perp}^2]}{3v_{ti}^2}\right)F_i\right]. \qquad \text{(A 10)}
\end{aligned}$$

Appendix B. Modified collision operators for the electron kinetic equation

The electron-electron collision operator is described in equation (A 1).

The modified ion-electron collision operator is

$$\mathcal{C}_{ei}[F_e, n_i, u_{i\|}, u_{e\|}, v_{te}](r, z, w_\|, w_\perp, t)$$
$$:= \frac{2\pi e^4 n_i \ln\Lambda}{(4\pi\epsilon_0)^2 m_e^2 v_{te}^3} \left\{ \frac{\partial}{\partial w_\|} \left[\mathcal{M}_{\|\|} \frac{\partial F_e}{\partial w_\|} + \mathcal{M}_{\|\perp} \frac{\partial F_e}{\partial w_\perp} + \left(1 + \frac{2(u_{i\|} - u_{e\|}) w_\|}{3 v_{te}} \right) \mathcal{F}_\| F_e \right] \right.$$
$$\left. + \frac{1}{w_\perp} \frac{\partial}{\partial w_\perp} \left[w_\perp \left(\mathcal{M}_{\|\perp} \frac{\partial F_e}{\partial w_\|} + \mathcal{M}_{\perp\perp} \frac{\partial F_e}{\partial w_\perp} + \frac{2(u_{i\|} - u_{e\|}) w_\perp}{3 v_{te}} \mathcal{F}_\| F_e \right) \right] \right\}, \quad \text{(B 1)}$$

where

$$\mathcal{M}_{\|\|}[u_{e\|}, v_{te}, u_{i\|}](r, z, w_\|, w_\perp, t) := \frac{w_\perp^2}{[(w_\| - (u_{i\|} - u_{e\|})/v_{te})^2 + w_\perp^2]^{3/2}}, \quad \text{(B 2)}$$

$$\mathcal{M}_{\|\perp}[u_{e\|}, v_{te}, u_{i\|}](r, z, w_\|, w_\perp, t) := -\frac{(w_\| - (u_{i\|} - u_{e\|})/v_{te})^2 w_\perp}{[(w_\| - (u_{i\|} - u_{e\|})/v_{te})^2 + w_\perp^2]^{3/2}}, \quad \text{(B 3)}$$

$$\mathcal{M}_{\perp\perp}[u_{e\|}, v_{te}, u_{i\|}](r, z, w_\|, w_\perp, t) := \frac{(w_\| - (u_{i\|} - u_{e\|})/v_{te})^2}{[(w_\| - (u_{i\|} - u_{e\|})/v_{te})^2 + w_\perp^2]^{3/2}} \quad \text{(B 4)}$$

and

$$\mathcal{F}_\|[F_e, u_{e\|}, v_{te}, u_{i\|}](r, z, t)$$
$$:= -4\pi \int_{-\infty}^{\infty} \mathrm{d}w_\| \int_0^\infty \mathrm{d}w_\perp \, \frac{w_\perp [w_\| - (u_{i\|} - u_{e\|})/v_{te}] F_e(z, w_\|, w_\perp, t)}{[(w_\| - (u_{i\|} - u_{e\|})/v_{te})^2 + w_\perp^2]^{3/2}}. \quad \text{(B 5)}$$

The modified electron-neutral collision operator is

$$\mathcal{C}_{en}[F_e, n_n, \mathbf{u}_n, u_{e\|}, v_{te}](z, w_\|, w_\perp, t) := -n_n R_{en} \left[F_e \right.$$
$$\left. - \frac{1}{8\pi^2} \int_0^\pi \mathrm{d}\chi \int_0^{2\pi} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\varphi' \, \sin\chi F_e(z, \overline{w}_\|, \overline{w}_\perp, t) \right]$$
$$+ n_n R_{en} \frac{\partial}{\partial w_\|} \left[\left(1 + \frac{2[(u_{n\|} - u_{e\|})^2 + u_{n\perp}^2] w_\|}{3 v_{te}^2} \right) F_e \right]$$
$$+ \frac{2 n_n R_{en} [(u_{n\|} - u_{e\|})^2 + u_{n\perp}^2]}{3 v_{te}^2 w_\perp} \frac{\partial}{\partial w_\perp} \left(w_\perp^2 F_e \right), \quad \text{(B 6)}$$

where

$$\overline{w}_\|[u_{e\|}, v_{te}, \mathbf{u}_n](r, z, w_\|, w_\perp, \chi, \varphi', t) := \frac{u_{n\|} - u_{e\|}}{v_{te}} + \overline{w}_{en} \cos\chi, \quad \text{(B 7)}$$

$$\overline{w}_\perp[u_{e\|}, v_{te}, u_{n\|}](r, z, w_\|, w_\perp, \chi, \varphi, \varphi' t) := \sqrt{\frac{u_{n\perp}^2}{v_{te}^2} + \overline{w}_{en}^2 \sin^2\chi - \frac{2 u_{n\perp} \overline{w}_{en}}{v_{te}} \sin\chi \cos\varphi} \quad \text{(B 8)}$$

and

$$\overline{w}_{en}[u_{e\|}, v_{te}, \mathbf{u}_n](r, z, w_\|, w_\perp, \varphi', t)$$
$$:= \sqrt{\left(w_\| + \frac{u_{e\|} - u_{n\|}}{v_{te}} \right)^2 + w_\perp^2 + \frac{u_{n\perp}^2}{v_{te}^2} - \frac{2 u_{n\perp} w_\perp}{v_{te}} \cos\varphi'}. \quad \text{(B 9)}$$

Appendix C. Modified collision operators for the neutral kinetic equation

The modified charge exchange collision operator for the neutral kinetic equation is

$$\begin{aligned}\mathcal{C}_{ni}[F_n, F_i, n_i, \mathbf{u}_n, u_{i\|}, v_{tn}, v_{ti}](r, z, w_r, w_z, w_\zeta, t) \\ := -\, n_i R_{in}\left[F_n - \frac{v_{tn}^3}{v_{ti}^3} F_i\left(r, z, \frac{u_{n\|} - u_{i\|}}{v_{ti}} + \frac{v_{tn}}{v_{ti}} w_\zeta, \frac{v_{tn}}{v_{ti}}\left|\mathbf{w}_\perp + \frac{\mathbf{u}_{n\perp}}{v_{tn}}\right|, t\right)\right] \\ + n_i R_{in} \nabla_w \cdot \left[\left(\frac{u_{i\|}\hat{\boldsymbol{\zeta}} - \mathbf{u}_n}{v_{tn}} + \frac{\mathbf{w}}{2}\left(\frac{v_{ti}^2}{v_{tn}^2} - 1 + \frac{2[(u_{n\|} - u_{i\|})^2 + u_{n\perp}^2]}{3v_{tn}^2}\right)\right) F_n\right],\end{aligned} \tag{C 1}$$

where

$$\left|\mathbf{w}_\perp + \frac{\mathbf{u}_{n\perp}}{v_{tn}}\right| \simeq \sqrt{\left(w_r + \frac{u_{nr}}{v_{tn}}\right)^2 + \left(w_z + \frac{u_{nz}}{v_{tn}}\right)^2}. \tag{C 2}$$

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