2D drift kinetic model with periodic boundary conditions

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1. Introduction

In report 2047357-TN-07-02, we presented a 2D drift kinetic model for a helical magnetic field with wall boundary conditions. The wall boundary conditions made it possible to obtain the potential and the electron flow from the conservation of parallel current and the electron parallel momentum equation. With periodic boundary conditions, appropriate for closed flux surfaces, it will not be sufficient with conservation of parallel current, and we will have to include terms small in the expansion parameter that we use, the ion gyroradius over the characteristic width of the scrape-off layer (ρ_i/L_r).

In this report, we first remind the reader of the content in report 2047357-TN-07-02, and we then explain how to modify the current conservation equations for cases with periodic boundary conditions.

2. Magnetic field, geometry and orderings

We use the cylindrical coordinates $\{r, z, \zeta\}$ (see report 2047357-TN-07-02 for the direction of increase of ζ). We consider a magnetized plasma with one ion species with charge e and mass m_i, electrons with charge $-e$ and mass m_e, and one species of neutrals with mass

$$m_n = m_i. \tag{2.1}$$

The plasma is magnetized by the helical magnetic field

$$\mathbf{B}(r, \zeta) := B_z(r)\hat{\mathbf{z}} + B_\zeta(r)\hat{\boldsymbol{\zeta}}(\zeta), \tag{2.2}$$

where $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\zeta}}$ are the unit vectors in the direction of ∇z and $\nabla\zeta$. Note that the components B_z and B_ζ only depend on the radial position r.

We assume that the plasma only varies in r and z. We assume that the electric field produced by the plasma is electrostatic, $\mathbf{E} = -(\partial\phi/\partial r)\hat{\mathbf{r}} - (\partial\phi/\partial z)\hat{\mathbf{z}}$, where $\hat{\mathbf{r}}$ is the unit vector in the direction ∇r. The potential $\phi(r, z, t)$ depends on the coordinates r and z and on time t.

We impose periodic boundary conditions at $z = 0$ and $z = L_z$. In the radial direction, we consider the interval between $r = r_0$ and $r = r_0 + L_r$. The length L_r is determined by a balance between the fast parallel velocity of the particles along magnetic field lines and their slow drift across them. The characteristic length between the two walls along a magnetic field line is of order

$$L_\parallel \sim \frac{B}{B_z} L_z. \tag{2.3}$$

Thus, the typical time that it takes for an ion to move from wall to wall is $L_\parallel/v_{ti} \sim$

$(B/B_z)(L_z/v_{ti})$, where $v_{ti} := \sqrt{2T_i/m_i}$ is the ion thermal speed and T_i is the ion temperature. For a potential ϕ of the order of T_i/e, where e is the proton charge, the radial $\mathbf{E}\times\mathbf{B}$ drift is

$$v_{Er} := -\frac{B_\zeta}{B^2}\frac{\partial\phi}{\partial z} \sim \frac{\rho_i}{L_z} v_{ti}, \tag{2.4}$$

where $\rho_i := v_{ti}/\Omega_i$ is the characteristic ion gyroradius and $\Omega_i := eB/m_i$ is the ion gyrofrequency. Thus, the time it takes for an ion to cross the domain in the radial direction is $L_r/v_{Er} \sim L_r L_z/\rho_i v_{ti}$. By making this time of the same order as $L_\parallel/v_{ti}$, we solve for L_r to find

$$L_r \sim \frac{B}{B_z}\rho_i \tag{2.5}$$

To simplify the problem to a tractable drift kinetic form, we assume that ρ_i is much smaller than L_r. This implies that

$$\frac{\rho_i}{L_r} \sim \frac{B_z}{B} \sim \frac{B_z}{B_\zeta} \ll 1, \tag{2.6}$$

that is, we will limit our model to magnetic fields that are mostly azimuthal and have a very small vertical component. This is an approximation that is consistent with magnetic field geometry in conventional tokamaks and also in the edge of many shots in spherical tokamaks.

We also assume that $r_0 \sim L_z \gg L_r$. Since r_0 is the characteristic length of variation of the magnetic field $\mathbf{B}$, the magnetic field barely changes across the domain $[r_0, r_0 + L_r]$. Thus, within our ordering, we assume $\mathbf{B}$ to be uniform in the domain of interest.

We assume the time derivatives to be of the same order as the parallel and perpendicular time scales that we have discussed above,

$$\frac{\partial}{\partial t} \sim \frac{\rho_i}{L_r}\frac{v_{ti}}{L_z}. \tag{2.7}$$

Our orderings above rest on the assumption $\phi \sim T_i/e$. In the case with wall boundary conditions, the wall boundary conditions ensured that ϕ remained of this order. With periodic boundary conditions, the size of ϕ is controlled by the momentum input. The force per unit volume on the plasma due to external sources, neutral-plasma collisions or ionization must satisfy

$$|\mathbf{F}_{i,\mathrm{ext}\perp}|, |\mathbf{F}_{e,\mathrm{ext}\perp}|, |\mathbf{F}_{in\perp}|, |\mathbf{F}_{en\perp}|, |\mathbf{F}_{i,\mathrm{ion}\perp}|, |\mathbf{F}_{e,\mathrm{ion}\perp}| \lesssim \left(\frac{\rho_i}{L_r}\right)^2 \frac{p_i}{L_z}, \tag{2.8}$$

where p_i is the ion pressure. This estimate for the force per unit volume comes from making the force of the order of the perpendicular inertia, $\partial(n_i m_i \mathbf{u}_{i\perp})/\partial t$, where the perpendicular ion velocity $\mathbf{u}_{i\perp}$ is taken to be of order $(\rho_i/L_r)v_{ti}$ (see equation (4.7) below for a justification of this ordering for $\mathbf{u}_{i\perp}$; also note that this means that the perpendicular flow is much smaller than the parallel one, which we assume to be of the order of v_{ti}). Equation (2.8) might seem stringent, but the friction between ions and neutrals and electron and neutrals (due to elastic collisions or ionization) is small in the closed field line region of the tokamak because the neutral density is small, i.e. we can assume that

$$n_n R_{in} \sim n_n R_{en}\sqrt{\frac{m_e}{m_i}} \sim n_n R_{\mathrm{ion}} \lesssim \left(\frac{\rho_i}{L_r}\right)^2 \frac{v_{ti}}{L_z}, \tag{2.9}$$

where $n_n R_{in}$, $n_n R_{en}$ and $n_n R_{\mathrm{ion}}$ are the ion-neutral, electron-neutral and ionization collision frequencies.

3. Summary of report 2047357-TN-07-02

The model in report 2047357-TN-07-02 is comprised of:

• three fluid equations (conservation of particles, parallel momentum and energy) for ions that have to be solved in conjunction with an ion kinetic equation to determine the ion density $n_i = n_e$, the ion parallel velocity $u_{i\|}$, the ion temperature T_i and the normalized ion distribution function F_i;

• five fluid equations (conservation of particles, the three components of momentum and energy) for neutrals that have to be solved in conjunction with a neutral kinetic equation to determine the neutral density n_n, the three components of the neutral velocity $\mathbf{u}_n$, the neutral temperature T_n and the normalized neutral distribution function F_n;

• two fluid equations (conservation of parallel current,

$$\frac{B_z}{B}\frac{\partial}{\partial z}\left[n_i\left(u_{i\|} - u_{e\|}\right)\right] = 0, \tag{3.1}$$

and conservation of energy) for electrons that have to be solved in conjunction with an electron kinetic equation to determine the electron parallel velocity $u_{e\|}$, the electron temperature T_e and the electron normalized distribution function F_e; and

• the electron parallel momentum equation,

$$0 = -\frac{B_z}{B}\frac{\partial p_{e\|}}{\partial z} + \frac{en_eB_z}{B}\frac{\partial \phi}{\partial z} + F_{ei\|} + n_e m_e n_n R_{en}(u_{n\|} - u_{e\|}), \tag{3.2}$$

for the potential ϕ. Here, $F_{ei\|}$ is the collisional friction force between electrons and ions.

In report 2047357-TN-07-02, we proposed a method to solve equations (3.1) and (3.2) in conjunction with wall boundary conditions. Equation (3.1) can be integrated in z to obtain $u_{e\|}(r,z,t) - u_{e\|}(r,0,t)$ (recall that n_i and $u_{i\|}$ are time-advanced using ion equations). With this result, equation (3.2) can be integrated in z to obtain the difference $\phi(r,z,t) - \phi(r,0,t)$ as a function of the unknown $u_{e\|}(r,0,t)$ (recall that $p_{e\|}$ is determined by the electron energy equation and the electron kinetic equation, and that $F_{ei\|}$ depends on $u_{e\|}$). With wall boundary conditions, we could solve for both $u_{e\|}(r,0,t)$ and $\phi(r,0,t)$. Unfortunately, the same cannot be said for periodic boundary conditions. With periodic boundary conditions and these equations, it is possible to find an equation for $u_{e\|}(r,0,t)$, but not for $\phi(r,0,t)$. Indeed, dividing equation (3.2) by n_e, integrating in z and using the periodic boundary conditions for ϕ, we find the condition

$$0 = \int_0^{L_z}\left[-\frac{B_z}{n_eB}\frac{\partial p_{e\|}}{\partial z} + \frac{F_{ei\|}}{n_e} + m_e n_n R_{en}(u_{n\|} - u_{e\|})\right]\mathrm{d}z. \tag{3.3}$$

This condition is satisfied by choosing the correct value of $u_{e\|}(r,0,t)$. Within equations (3.1) and (3.2), there is no other similar condition for $\phi(r,0,t)$. To find such a condition, we need to modify equation (3.1) by keeping higher order terms in ρ_i/L_r.

4. Current conservation for periodic boundary conditions

Parra & Catto (2008) showed that gyrokinetic equations (and consequently the subsidiary limit of drift kinetics) require higher order terms in ρ_i/L_r in order to determine the component of the electric field perpendicular to the flux surfaces traced by the magnetic field when the magnetic field is axisymmetric. In this case, this means that one needs higher order terms to determine $\partial\phi/\partial r$.

The need for higher order terms can be demonstrated using moments of the full kinetic equation. Before we start taking moments, we explain what we mean by full kinetic distribution function and full kinetic equation in this report, and how they compare to the drift kinetic distribution function and kinetic equation that we have used so far. Importantly, what follows does not apply to the neutral distribution function and kinetic equation as we have not reduced them using the drift kinetic approximation in previous reports.

We denote the ion and electron full distribution functions by $g_s(r, z, \mathbf{v}, t)$, where the velocity is given by

$$\mathbf{v} = v_\parallel \hat{\mathbf{b}} + \mathbf{v}_\perp = v_\parallel \hat{\mathbf{b}} + v_\perp (\cos\varphi\, \hat{\mathbf{r}} + \sin\varphi\, \hat{\mathbf{b}} \times \hat{\mathbf{r}}), \tag{4.1}$$

Here $v_\parallel$ is the velocity parallel to the magnetic field, $\hat{\mathbf{b}} := \mathbf{B}/B$ is the unit vector in the direction to the magnetic field, and the velocity perpendicular to the magnetic field $\mathbf{v}_\perp$ is described by its magnitude $v_\perp$ and its direction, given by the gyrophase φ. The full kinetic distribution functions for ions and electrons, g_i and g_e, satisfy the full kinetic equations

$$\frac{\partial g_i}{\partial t} + \nabla \cdot (\mathbf{v} g_i) + \nabla_v \cdot \left[\frac{e}{m_i} (-\nabla\phi + \mathbf{v} \times \mathbf{B})\, g_i \right] = C_{ii} + C_{in} + C_{ie} + C_{i,\mathrm{ion}} + Q_i \tag{4.2}$$

and

$$\frac{\partial g_e}{\partial t} + \nabla \cdot (\mathbf{v} g_e) + \nabla_v \cdot \left[-\frac{e}{m_e} (-\nabla\phi + \mathbf{v} \times \mathbf{B})\, g_e \right] = C_{ee} + C_{en} + C_{ei} + C_{e,\mathrm{ion}} + Q_e. \tag{4.3}$$

Note that we have included like-particle collisions, electron-ion collisions, collisions with neutrals and ionization collisions. The sources Q_e represent the particle, momentum and energy input into the plasma.

The drift kinetic ion and electron distribution functions f_i and f_e that we used in previous reports are gyroaverages of the full distribution functions,

$$f_s(r, z, v_\parallel, v_\perp, t) := \frac{1}{2\pi} \int_0^{2\pi} g_s(r, z, \mathbf{v}(v_\parallel, v_\perp, \varphi), t)\, \mathrm{d}\varphi. \tag{4.4}$$

(Correspondingly, the sources S_s that we used are the gyoraverage of the full sources Q_s.) For $\rho_i/L_r \ll 1$, the distribution functions g_s and f_s are almost identical, $g_s \simeq f_s$ (Hazeltine 1973). The difference between these distribution functions, of order $\rho_i/L_r \ll 1$, was negligible in the case with wall boundary conditions, but it will be important with periodic boundary conditions. Thus, we write

$$g_s = f_s + g_{s1} + \ldots, \tag{4.5}$$

where, neglecting the ion-neutral, electron-neutral and ionization collisions to lowest order (recall equation (2.9)),

$$g_{s1} = \sin\varphi \left(\frac{v_\perp}{\Omega_s} \frac{\partial f_s}{\partial r} - \frac{1}{B} \frac{\partial \phi}{\partial r} \frac{\partial f_s}{\partial v_\perp} \right) \sim \frac{\rho_s}{L_r} f_s. \tag{4.6}$$

Here $\Omega_s := Z_s e B/m_s$ is the gyrofrequency of species s, and Z_s is the charge number (1 for ions and -1 for electrons). Using equation (4.5), we can calculate the perpendicular average velocity of ions and electrons,

$$\mathbf{u}_{s\perp} := \frac{1}{n_s} \int \mathbf{v}_\perp g_s\, \mathrm{d}^3 v \simeq \frac{1}{n_s} \int \mathbf{v}_\perp g_{s1}\, \mathrm{d}^3 v = \hat{\mathbf{b}} \times \hat{\mathbf{r}} \left(\frac{1}{Z_s e n_s B} \frac{\partial p_{s\perp}}{\partial r} + \frac{1}{B} \frac{\partial \phi}{\partial r} \right) \sim \frac{\rho_i}{L_r} v_{ti}, \tag{4.7}$$

where $p_{s\perp} := \int (m_s v_\perp^2/2) f_s\, \mathrm{d}^3 v$ is the perpendicular pressure of species s. Note that since the average of $\mathbf{v}_\perp$ over the gyrophase vanishes, $\mathbf{u}_{s\perp}$ is only due to the small correction g_{s1}. Thus, $\mathbf{u}_{s\perp}$ is small. Note that this is a consequence of our assumption $\phi \sim T_i/e$. A larger ϕ would have led to a larger average velocity.

After this brief introduction to the full distribution functions g_s, we can use moments of these distribution functions to impose current conservation following the procedure proposed by Parra & Catto (2009). Current conservation can be written as

$$\nabla \cdot \left[e n_i (u_{i\|} - u_{e\|}) \hat{\mathbf{b}} \right] + \nabla \cdot \mathbf{J}_\perp = 0, \tag{4.8}$$

where $\mathbf{J}_\perp$ is the component of the current density perpendicular to the magnetic field. In report 2047357-TN-07-02 we could neglect the term $\nabla \cdot \mathbf{J}_\perp$ because it is small in $L_r/L_z \ll 1$. We cannot neglect it any longer, as we need it to determine $\phi(r, 0, t)$. Equation (4.8) can be written as

$$\frac{\partial}{\partial z} \left[\frac{e n_i B_z}{B} (u_{i\|} - u_{e\|}) + \mathbf{J}_\perp \cdot \hat{\mathbf{z}} \right] + \frac{1}{r} \frac{\partial}{\partial r} \left(r \mathbf{J}_\perp \cdot \hat{\mathbf{r}} \right) = 0. \tag{4.9}$$

Due to the periodic boundary conditions, the large parallel current term can be eliminated by integrating in z,

$$\frac{\partial}{\partial r} \left(\int_0^{L_z} \mathbf{J}_\perp \cdot \hat{\mathbf{r}} \, \mathrm{d}z \right) = 0, \tag{4.10}$$

where we have also used the fact that $r \simeq r_0$ is almost constant. Condition (4.10) determines $\phi(r, 0, t)$.

Unfortunately, equation (4.7) is not enough to compute $\mathbf{J}_\perp \cdot \hat{\mathbf{r}}$ to the sufficiently high order needed to obtain $\phi(r, 0, t)$. The perpendicular current density can, however, be calculated to very high order by taking moments of the full kinetic equations for ions and electrons. Multiplying equations (4.2) and (4.3) by $m_s \mathbf{v}$ and integrating over velocity space, we find the ion and electron total momentum equations,

$$\frac{\partial}{\partial t}(n_i m_i \mathbf{u}_i) + \nabla \cdot \left(\int m_i \mathbf{v}\mathbf{v} g_i \, \mathrm{d}^3v \right) = -e n_i \nabla\phi + e n_i \mathbf{u}_i \times \mathbf{B} + \mathbf{F}_{in} + \mathbf{F}_{ie} + \mathbf{F}_{i,\mathrm{ion}} + \mathbf{F}_{i,\mathrm{ext}} \tag{4.11}$$

and

$$\frac{\partial}{\partial t}(n_e m_e \mathbf{u}_e) + \nabla \cdot \left(\int m_e \mathbf{v}\mathbf{v} g_e \, \mathrm{d}^3v \right) = e n_e \nabla\phi - e n_e \mathbf{u}_e \times \mathbf{B} + \mathbf{F}_{en} + \mathbf{F}_{ei} + \mathbf{F}_{e,\mathrm{ion}} + \mathbf{F}_{e,\mathrm{ext}}. \tag{4.12}$$

Here, $\mathbf{F}_\alpha := \int m_s \mathbf{v} C_\alpha \, \mathrm{d}^3v$ is the force due to the collision operator C_α, and $\mathbf{F}_{s,\mathrm{ext}} := \int m_s \mathbf{v} Q_s \, \mathrm{d}^3v$ is the external momentum input. Summing equations (4.11) and (4.12), using quasineutrality and $\mathbf{F}_{ie} + \mathbf{F}_{ei} = 0$, and neglecting the electron momentum compared to the ion momentum, we find

$$\mathbf{J} \times \mathbf{B} \simeq \nabla \cdot \left(\int m_i \mathbf{v}\mathbf{v} g_i \, \mathrm{d}^3v + \int m_e \mathbf{v}\mathbf{v} g_e \, \mathrm{d}^3v \right) + \frac{\partial}{\partial t}(n_i m_i \mathbf{u}_i) - \mathbf{F}_{in} - \mathbf{F}_{i,\mathrm{ion}} - \mathbf{F}_{\mathrm{ext}}. \tag{4.13}$$

where we have defined the total external force $\mathbf{F}_{\mathrm{ext}} := \mathbf{F}_{i,\mathrm{ext}} + \mathbf{F}_{e,\mathrm{ext}}$. Multiplying equation (4.13) by $(\hat{\mathbf{r}} \times \hat{\mathbf{b}})/B$, we obtain the radial component of $\mathbf{J}_\perp$,

$$\begin{aligned} \mathbf{J}_\perp \cdot \hat{\mathbf{r}} \simeq & -\frac{1}{B} \nabla \cdot \left(\int m_i \mathbf{v}\mathbf{v} \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{r}}) g_i \, \mathrm{d}^3v + \int m_e \mathbf{v} \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{r}}) g_e \, \mathrm{d}^3v \right) \\ & + \frac{1}{B} \int m_i \mathbf{v} \cdot \nabla \left(\hat{\mathbf{b}} \times \hat{\mathbf{r}} \right) \cdot \mathbf{v} g_i \, \mathrm{d}^3v + \frac{1}{B} \int m_e \mathbf{v} \cdot \nabla \left(\hat{\mathbf{b}} \times \hat{\mathbf{r}} \right) \cdot \mathbf{v} g_e \, \mathrm{d}^3v \\ & - \frac{\partial}{\partial t} \left[\frac{n_i m_i}{B} \mathbf{u}_i \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{r}}) \right] + \frac{1}{B} \mathbf{F}_{in} \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{r}}) + \frac{1}{B} \mathbf{F}_{i,\mathrm{ion}} \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{r}}) + \frac{1}{B} \mathbf{F}_{\mathrm{ext}} \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{r}}), \end{aligned} \tag{4.14}$$

where

$$\nabla\left(\hat{\mathbf{b}}\times\hat{\mathbf{r}}\right) = -\frac{B_\zeta^2}{B^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B_z}{B_\zeta}\right)\hat{\mathbf{r}}\hat{\mathbf{b}} - \frac{B_z B_\zeta}{rB^2}\hat{\mathbf{b}}\hat{\mathbf{r}} + \frac{B_z^2}{rB^2}(\hat{\mathbf{b}}\times\hat{\mathbf{r}})\hat{\mathbf{r}} \sim \frac{B_z}{r_0 B} \sim \frac{\rho_i}{L_r}\frac{1}{L_z}. \tag{4.15}$$

The term with the time derivative in the right side of equation (4.14) sets the size of the equation. For $\partial/\partial t \sim \rho_i v_{ti}/L_r L_z$, using the fact that $|\mathbf{u}_{i\perp}| \sim (\rho_i/L_r)v_{ti}$, we find

$$\frac{\partial}{\partial t}\left[\frac{n_i m_i}{B}\mathbf{u}_i\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{r}})\right] \sim \left(\frac{\rho_i}{L_r}\right)^2\frac{p_i}{BL_z}. \tag{4.16}$$

This estimate justifies the ordering (2.8). Also, estimate (4.16) sets the size of the terms that we need to keep in the equation. For example, the integrals that contain $\nabla(\hat{\mathbf{b}}\times\hat{\mathbf{r}})$ are small because $\nabla(\hat{\mathbf{b}}\times\hat{\mathbf{r}})$ is itself small in ρ_i/L_r and the integrals over velocity are small by a factor of $(\rho_i/L_r)^2$, as can be checked using expression (4.6) for g_i and g_e. Thus, integrating over z, using equation (4.7) for $\mathbf{u}_{i\perp}$ and employing the approximation $r \simeq r_0$, we find

$$\begin{aligned}\int_0^{L_z}\mathbf{J}_\perp\cdot\hat{\mathbf{r}}\,\mathrm{d}z \simeq{}& -\frac{\partial}{\partial t}\left[\int_0^{L_z}\frac{n_i m_i}{B^2}\left(\frac{\partial\phi}{\partial r}+\frac{1}{en_i}\frac{\partial p_{i\perp}}{\partial r}\right)\mathrm{d}z\right]\\ &-\frac{1}{B}\frac{\partial}{\partial r}\left[\int_0^{L_z}\left(\int m_i\mathbf{v}\cdot\hat{\mathbf{r}}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\hat{\mathbf{r}}\,g_i\,\mathrm{d}^3v+\int m_e\mathbf{v}\cdot\hat{\mathbf{r}}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\hat{\mathbf{r}}\,g_e\,\mathrm{d}^3v\right)\mathrm{d}z\right]\\ &+\int_0^{L_z}\left[\frac{1}{B}\mathbf{F}_{in}\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{r}})+\frac{1}{B}\mathbf{F}_{i,\mathrm{ion}}\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{r}})+\frac{1}{B}\mathbf{F}_{\mathrm{ext}}\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{r}})\right]\mathrm{d}z.\end{aligned} \tag{4.17}$$

The velocity integrals in equation (4.17) can be calculated using a moment of equations (4.2) and (4.3). Multiplying equation (4.2) by $(m_i^2/4eB)[(\mathbf{v}\cdot\hat{\mathbf{r}})^2 - ((\mathbf{v}\times\hat{\mathbf{b}})\cdot\hat{\mathbf{r}})^2]$ and integrating in velocity space and z, we find

$$\begin{aligned}&\int_0^{L_z}\mathrm{d}z\int\mathrm{d}^3v\,m_i\mathbf{v}\cdot\hat{\mathbf{r}}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\hat{\mathbf{r}}\,g_i\\ &\quad\simeq\frac{1}{4\Omega_i}\frac{\partial}{\partial t}\left[\int_0^{L_z}\mathrm{d}z\int\mathrm{d}^3v\,m_i[(\mathbf{v}\cdot\hat{\mathbf{r}})^2-((\mathbf{v}\times\hat{\mathbf{b}})\cdot\hat{\mathbf{r}})^2]\,g_i\right]\\ &\quad+\frac{1}{4\Omega_i}\frac{\partial}{\partial r}\left[\int_0^{L_z}\mathrm{d}z\int\mathrm{d}^3v\,m_i\mathbf{v}\cdot\hat{\mathbf{r}}[(\mathbf{v}\cdot\hat{\mathbf{r}})^2-((\mathbf{v}\times\hat{\mathbf{b}})\cdot\hat{\mathbf{r}})^2]\,g_i\right]\\ &\quad-\int_0^{L_z}\frac{n_i m_i}{2B^2}\left[\frac{\partial\phi}{\partial r}\left(\frac{\partial\phi}{\partial z}+\frac{1}{en_i}\frac{\partial p_{i\perp}}{\partial z}\right)+\frac{\partial\phi}{\partial z}\left(\frac{\partial\phi}{\partial r}+\frac{1}{en_i}\frac{\partial p_{i\perp}}{\partial r}\right)\right]\mathrm{d}z\\ &\quad-\frac{1}{4\Omega_i}\int_0^{L_z}\mathrm{d}z\int\mathrm{d}^3v\,m_i[(\mathbf{v}\cdot\hat{\mathbf{r}})^2-((\mathbf{v}\times\hat{\mathbf{b}})\cdot\hat{\mathbf{r}})^2]C_{ii},\end{aligned} \tag{4.18}$$

where we have neglected most collision operators due to our assumption (2.9). Due to the integrals over the gyrophase, the terms with the time derivative and the ion-ion collision operator vanish to the order of interest, leaving

$$
\begin{aligned}
\int_0^{L_z} \mathrm{d}z \int \mathrm{d}^3 v \, m_i \mathbf{v} \cdot \hat{\mathbf{r}} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \hat{\mathbf{r}} \, g_i \\
\simeq \frac{1}{4\Omega_i} \frac{\partial}{\partial r} \left[\int_0^{L_z} \mathrm{d}z \int \mathrm{d}^3 v \, m_i \mathbf{v} \cdot \hat{\mathbf{r}} [(\mathbf{v} \cdot \hat{\mathbf{r}})^2 - ((\mathbf{v} \times \hat{\mathbf{b}}) \cdot \hat{\mathbf{r}})^2] \, g_i \right] \\
- \int_0^{L_z} \frac{n_i m_i}{2B^2} \left[\frac{\partial \phi}{\partial r} \left(\frac{\partial \phi}{\partial z} + \frac{1}{e n_i} \frac{\partial p_{i\perp}}{\partial z} \right) + \frac{\partial \phi}{\partial z} \left(\frac{\partial \phi}{\partial r} + \frac{1}{e n_i} \frac{\partial p_{i\perp}}{\partial r} \right) \right] \mathrm{d}z. \qquad (4.19)
\end{aligned}
$$

There is still another integral over velocity to be calculated. We can use the moment $(m_i^2/3eB)[((\mathbf{v} \times \hat{\mathbf{b}}) \cdot \hat{\mathbf{r}})^3 + 3(\mathbf{v} \cdot \hat{\mathbf{r}})^2 (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \hat{\mathbf{r}}]$ of equation (4.2) to find

$$
\int_0^{L_z} \mathrm{d}z \int \mathrm{d}^3 v \, m_i \mathbf{v} \cdot \hat{\mathbf{r}} [(\mathbf{v} \cdot \hat{\mathbf{r}})^2 - ((\mathbf{v} \times \hat{\mathbf{b}}) \cdot \hat{\mathbf{r}})^2] \, g_i \simeq - \int_0^{L_z} \frac{2 p_{i\perp}}{B} \frac{\partial \phi}{\partial z} \mathrm{d}z. \qquad (4.20)
$$

With this result and employing integration by parts in z to write

$$
\begin{aligned}
\frac{\partial}{\partial r} \left[\int_0^{L_z} \frac{2 p_{i\perp}}{B} \frac{\partial \phi}{\partial z} \mathrm{d}z \right] &\simeq \int_0^{L_z} \frac{2}{B} \left(\frac{\partial p_{i\perp}}{\partial r} \frac{\partial \phi}{\partial z} + p_{i\perp} \frac{\partial^2 \phi}{\partial r \partial z} \right) \mathrm{d}z \\
&= \int_0^{L_z} \frac{2}{B} \left(\frac{\partial p_{i\perp}}{\partial r} \frac{\partial \phi}{\partial z} - \frac{\partial p_{i\perp}}{\partial z} \frac{\partial \phi}{\partial r} \right) \mathrm{d}z, \qquad (4.21)
\end{aligned}
$$

equation (4.19) simplifies to

$$
\int_0^{L_z} \mathrm{d}z \int \mathrm{d}^3 v \, m_i \mathbf{v} \cdot \hat{\mathbf{r}} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \hat{\mathbf{r}} \, g_i = - \int_0^{L_z} \frac{n_i m_i}{B^2} \frac{\partial \phi}{\partial z} \left(\frac{\partial \phi}{\partial r} + \frac{1}{e n_i} \frac{\partial p_{i\perp}}{\partial r} \right) \mathrm{d}z. \qquad (4.22)
$$

The electron integral in equation (4.17) can be calculated using the same procedure that has led to equation (4.22), and it turns out to be negligible due to the smallness of m_e/m_i.

Substituting equation (4.22) into equation (4.17) and then equation (4.17) into equation (4.10), and using the same simplified ion-neutral and ionization collision operators that we have used in previous reports,

$$
C_{in} := -R_{in}(n_n g_i - n_i f_n), \quad C_{i,\mathrm{ion}} := n_e R_{\mathrm{ion}} f_n, \qquad (4.23)
$$

we find the final equation for $\phi(r, 0, t)$,

$$
\begin{aligned}
\frac{\partial}{\partial r} \Bigg\{ & - \frac{\partial}{\partial t} \int_0^{L_z} \left[\frac{n_i m_i}{B^2} \left(\frac{\partial \phi}{\partial r} + \frac{1}{e n_i} \frac{\partial p_{i\perp}}{\partial r} \right) \right] \mathrm{d}z \\
& + \frac{1}{B} \frac{\partial}{\partial r} \left[\int_0^{L_z} \frac{n_i m_i}{B^2} \frac{\partial \phi}{\partial z} \left(\frac{\partial \phi}{\partial r} + \frac{1}{e n_i} \frac{\partial p_{i\perp}}{\partial r} \right) \mathrm{d}z \right] \\
& - \int_0^{L_z} \frac{n_i m_i n_n R_{in}}{B^2} \left(\frac{\partial \phi}{\partial r} + \frac{1}{e n_i} \frac{\partial p_{i\perp}}{\partial r} \right) \mathrm{d}z \\
& + \int_0^{L_z} \frac{n_i m_i n_n (R_{in} + R_{\mathrm{ion}}) u_{nz}}{B} \mathrm{d}z + \int_0^{L_z} \frac{\mathbf{F}_{\mathrm{ext}} \cdot \hat{\mathbf{z}}}{B} \mathrm{d}z \Bigg\} = 0. \qquad (4.24)
\end{aligned}
$$

Note that in several places we have used the approximation $\hat{\mathbf{b}} \times \hat{\mathbf{r}} \simeq \hat{\mathbf{z}}$.

Equation (4.24) is small by a factor of $(\rho_i/L_r)2$ compared to the parallel current term in the current conservation equation (4.8), of order $(\rho_i/L_r)(e n_e v_{ti}/L_z)$. Thus, in principle, to keep these terms we would need to keep terms up to order $(\rho_i/L_r)^3$ in this equation. Thankfully, we do not need to do that because we now have equation (4.24) in explicit form. We can add equation (4.24) to our set of equations in a consistent manner.

We finish by relating equation (4.24) to momentum conservation, a connection that we mentioned at the end of section 2. Equation (4.24) is a radial derivative of $\int_0^{L_z} \mathbf{J}_\perp \cdot \hat{\mathbf{r}}\,\mathrm{d}z$ (see equation (4.10)). Then, equation (4.24) can be integrated to show that $\int_0^{L_z} \mathbf{J}_\perp \cdot \hat{\mathbf{r}}\,\mathrm{d}z$ is a constant that we will call I_r, giving

$$\frac{\partial}{\partial t}\int_0^{L_z} n_i m_i \mathbf{u}_{i\perp} \cdot \hat{\mathbf{z}}\,\mathrm{d}z - \frac{\partial}{\partial r}\left(\int_0^{L_z} \frac{n_i m_i}{B}\frac{\partial \phi}{\partial z}\mathbf{u}_{i\perp} \cdot \hat{\mathbf{z}}\,\mathrm{d}z\right) = -I_r B$$
$$- \int_0^{L_z} n_i m_i n_n R_{in}(\mathbf{u}_{i\perp} \cdot \hat{\mathbf{z}} - u_{nz})\,\mathrm{d}z$$
$$+ \int_0^{L_z} n_i m_i n_n R_{\mathrm{ion}} u_{nz}\,\mathrm{d}z + \int_0^{L_z} \mathbf{F}_{\mathrm{ext}} \cdot \hat{\mathbf{z}}\,\mathrm{d}z. \tag{4.25}$$

Here we have used the $\hat{\mathbf{z}}$ projection of equation (4.7) to rewrite $\partial\phi/\partial r$ as a function of $\mathbf{u}_{i\perp} \cdot \hat{\mathbf{z}}$. Equation (4.25) is the perpendicular momentum balance in the z direction, and it includes the magnetic force due to the radial current I_r. This magnetic force is zero in tokamaks, where the radial current vanishes.

5. Moment drift kinetics

The moment drift kinetic formulation of the problem with periodic boundary conditions is thus the equations in report 2047357-TN-07-02 plus our new equation (4.24) for $\phi(r, 0, t)$. The perpendicular pressure appearing in this equation can be calculated from the normalized distribution function using the formula

$$p_{s\perp}[n_s, v_{ts}, F_s](r, z) := \pi n_s m_s v_{ts}^2 \int_{-\infty}^{\infty} \mathrm{d}w_\parallel \int_0^\infty \mathrm{d}w_\perp\, w_\perp^3 F_s(r, z, w_\parallel, w_\perp, t). \tag{5.1}$$

6. Discussion

Note that the addition of equation (4.24) has only been possible because we evolve the densities of ions and electrons independently of their normalized distribution functions. Had we proposed to evolve the unnormalized ion and electron distribution functions, it would not have been possible to have an independent higher-order current conservation equation because it would not be consistent with the density that arises from the time evolution of the lowest order kinetic equations.

When implementing the equations proposed in this report, it is important to ensure that there is z-variation of density, temperature and flows. Otherwise, the equations become trivial. When connected to the open field line region (the topic of a future report), the z-variation will arise naturally due to the wall boundary conditions. In the absence of open field lines, one can use sources and sinks with that are not uniform in z (i.e. excess ionization in the region close to the divertor, where most neutrals are).

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