# 2D drift kinetics in a helical field with a 'separatrix'

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## 1. Introduction

In reports 2047357-TN-07-02 and 2047357-TN-09-01, we presented 2D drift kinetic models with wall boundary conditions and periodic boundary conditions, respectively. The difference between the two models is significant: with periodic boundary conditions, we needed to add another equation to determine a piece of the potential that only depends on the radial coordinate. In this report, we discuss how one would connect both models if one considers a situation in which he boundary conditions switch from being periodic to being wall boundary conditions at some radial position $r = r_s$. The position $r = r_s$ is an effective 'separatrix'.

In this report, we first remind the reader of the content in reports 2047357-TN-07-02 and 2047357-TN-09-01, and we then explain how these models connect at $r = r_s$. Remarkably, the connection between these two types of boundary conditions requires a full gyrokinetic treatment around the separatrix. Such a treatment is out of scope of the ExCALIBUR contract, focused on drift kinetics, so we propose solving the equations with a 'separatrix' in a simple limit in which the treatment can be approximated with a drift kinetic model.

## 2. Magnetic field, geometry and orderings

We use the cylindrical coordinates $\{r, z, \zeta\}$ (see report 2047357-TN-07-02 for the direction of increase of $\zeta$). We consider a magnetized plasma with one ion species with charge $e$ and mass $m_i$, electrons with charge $-e$ and mass $m_e$, and one species of neutrals with mass

$$m_n = m_i. \tag{2.1}$$

The plasma is magnetized by the helical magnetic field

$$\mathbf{B}(r,\zeta) := B_z(r)\hat{\mathbf{z}} + B_\zeta(r)\hat{\boldsymbol{\zeta}}(\zeta), \tag{2.2}$$

where $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\zeta}}$ are the unit vectors in the direction of $\nabla z$ and $\nabla \zeta$. Note that the components $B_z$ and $B_\zeta$ only depend on the radial position $r$.

We assume that the plasma only varies in $r$ and $z$. We assume that the electric field produced by the plasma is electrostatic, $\mathbf{E} = -(\partial\phi/\partial r)\hat{\mathbf{r}} - (\partial\phi/\partial z)\hat{\mathbf{z}}$, where $\hat{\mathbf{r}}$ is the unit vector in the direction $\nabla r$. The potential $\phi(r, z, t)$ depends on the coordinates $r$ and $z$ and on time $t$.

We impose periodic or wall boundary conditions at $z = 0$ and $z = L_z$. In the radial direction, we consider the interval between $r = r_0$ and $r = r_0 + L_r$. The length $L_r$ is determined by a balance between the fast parallel velocity of the particles along magnetic field lines and their slow drift across them. The characteristic length between the two walls along a magnetic field line is of order

$$L_\parallel \sim \frac{B}{B_z} L_z. \tag{2.3}$$

Thus, the typical time that it takes for an ion to move from wall to wall is $L_\parallel/v_{ti} \sim (B/B_z)(L_z/v_{ti})$, where $v_{ti} := \sqrt{2T_i/m_i}$ is the ion thermal speed and $T_i$ is the ion temperature. For a potential $\phi$ of the order of $T_i/e$, where $e$ is the proton charge, the radial $\mathbf{E} \times \mathbf{B}$ drift is

$$v_{Er} := -\frac{B_\zeta}{B^2} \frac{\partial \phi}{\partial z} \sim \frac{\rho_i}{L_z} v_{ti}, \tag{2.4}$$

where $\rho_i := v_{ti}/\Omega_i$ is the characteristic ion gyroradius and $\Omega_i := eB/m_i$ is the ion gyrofrequency. Thus, the time it takes for an ion to cross the domain in the radial direction is $L_r/v_{Er} \sim L_r L_z/\rho_i v_{ti}$. By making this time of the same order as $L_\parallel/v_{ti}$, we solve for $L_r$ to find

$$L_r \sim \frac{B}{B_z} \rho_i \tag{2.5}$$

To simplify the problem to a tractable drift kinetic form, we assume that $\rho_i$ is much smaller than $L_r$. This implies that

$$\frac{\rho_i}{L_r} \sim \frac{B_z}{B} \simeq \frac{B_z}{B_\zeta} \ll 1, \tag{2.6}$$

that is, we will limit our model to magnetic fields that are mostly azimuthal and have a very small vertical component. This is an approximation that is consistent with magnetic field geometry in conventional tokamaks and also in the edge of many shots in spherical tokamaks.

We also assume that $r_0 \sim L_z \gg L_r$. Since $r_0$ is the characteristic length of variation of the magnetic field $\mathbf{B}$, the magnetic field barely changes across the domain $[r_0, r_0 + L_r]$. Thus, within our ordering, we assume $\mathbf{B}$ to be uniform in the domain of interest.

We assume the time derivatives to be of the same order as the parallel and perpendicular time scales that we have discussed above,

$$\frac{\partial}{\partial t} \sim \frac{\rho_i}{L_r} \frac{v_{ti}}{L_z}. \tag{2.7}$$

Our orderings above are based on the assumption $\phi \sim T_i/e$. In the region of space with wall boundary conditions, the wall boundary conditions ensure that $\phi$ remains of this order. With periodic boundary conditions, the size of $\phi$ is controlled by the momentum input. The force per unit volume on the plasma due to external sources, neutral-plasma collisions or ionization must satisfy

$$|\mathbf{F}_{i,\mathrm{ext}\perp}|, |\mathbf{F}_{e,\mathrm{ext}\perp}|, |\mathbf{F}_{in\perp}|, |\mathbf{F}_{en\perp}|, |\mathbf{F}_{i,\mathrm{ion}\perp}|, |\mathbf{F}_{e,\mathrm{ion}\perp}| \lesssim \left(\frac{\rho_i}{L_r}\right)^2 \frac{p_i}{L_z}, \tag{2.8}$$

where $p_i$ is the ion pressure. This estimate for the force per unit volume comes from making the force of the order of the perpendicular inertia, $\partial(n_i m_i \mathbf{u}_{i\perp})/\partial t$, where the perpendicular ion velocity $\mathbf{u}_{i\perp}$ is taken to be of order $(\rho_i/L_r)v_{ti}$ (note that this means that the perpendicular flow is much smaller than the parallel one, which we assume to be of the order of $v_{ti}$). Equation (2.8) might seem stringent, but the friction between ions and neutrals and electron and neutrals (due to elastic collisions or ionization) is small in the closed field line region of the tokamak because the neutral density is small, i.e. we can assume that

$$n_n R_{in} \sim n_n R_{en} \sqrt{\frac{m_e}{m_i}} \sim n_n R_{\text{ion}} \lesssim \left(\frac{\rho_i}{L_r}\right)^2 \frac{v_{ti}}{L_z}, \tag{2.9}$$

where $n_n R_{in}$, $n_n R_{en}$ and $n_n R_{\text{ion}}$ are the ion-neutral, electron-neutral and ionization collision frequencies.

## 3. Summary of reports 2047357-TN-07-02 and 2047357-TN-09-01

The models in reports 2047357-TN-07-02 and 2047357-TN-09-01 are comprised of:

- three fluid equations (conservation of particles, parallel momentum and energy) for ions that have to be solved in conjunction with an ion kinetic equation to determine the ion density $n_i = n_e$, the ion parallel velocity $u_{i\|}$, the ion temperature $T_i$ and the normalized ion distribution function $F_i$;
- five fluid equations (conservation of particles, the three components of momentum and energy) for neutrals that have to be solved in conjunction with a neutral kinetic equation to determine the neutral density $n_n$, the three components of the neutral velocity $\mathbf{u}_n$, the neutral temperature $T_n$ and the normalized neutral distribution function $F_n$;
- two fluid equations (conservation of parallel current,

$$\frac{B_z}{B}\frac{\partial}{\partial z}\left[n_i\left(u_{i\|} - u_{e\|}\right)\right] = 0, \tag{3.1}$$

and conservation of energy) for electrons that have to be solved in conjunction with an electron kinetic equation to determine the electron parallel velocity $u_{e\|}$, the electron temperature $T_e$ and the electron normalized distribution function $F_e$; and
- the electron parallel momentum equation,

$$0 = -\frac{B_z}{B}\frac{\partial p_{e\|}}{\partial z} + \frac{e n_e B_z}{B}\frac{\partial \phi}{\partial z} + F_{ei\|} + n_e m_e n_n R_{en}(u_{n\|} - u_{e\|}), \tag{3.2}$$

for the potential $\phi$. Here, $F_{ei\|}$ is the collisional friction force between electrons and ions.

In report 2047357-TN-07-02, we proposed a method to solve equations (3.1) and (3.2) in conjunction with wall boundary conditions. Equation (3.1) can be integrated in $z$ to obtain $u_{e\|}(r,z,t) - u_{e\|}(r,0,t)$ (recall that $n_i$ and $u_{i\|}$ are time-advanced using ion equations). With this result, equation (3.2) can be integrated in $z$ to obtain the difference $\phi(r,z,t) - \phi(r,0,t)$ as a function of the unknown $u_{e\|}(r,0,t)$ (recall that $p_{e\|}$ is determined by the electron energy equation and the electron kinetic equation, and that $F_{ei\|}$ depends on $u_{e\|}$). With wall boundary conditions, we could solve for both $u_{e\|}(r,0,t)$ and $\phi(r,0,t)$.

In report 2047357-TN-09-01, we explained how to use equations (3.1) and (3.2) with periodic boundary conditions. As in the case with wall boundary conditions, equations (3.1) and (3.2) give $u_{e\|}(r,z,t) - u_{e\|}(r,0,t)$ and $\phi(r,z,t) - \phi(r,0,t)$ as functions of the unknown $u_{e\|}(r,0,t)$. Dividing equation (3.2) by $n_e$, integrating in $z$ and using the periodic boundary conditions for $\phi$, we find the condition

$$0 = \int_0^{L_z}\left[-\frac{B_z}{n_e B}\frac{\partial p_{e\|}}{\partial z} + \frac{F_{ei\|}}{n_e} + m_e n_n R_{en}(u_{n\|} - u_{e\|})\right]\mathrm{d}z. \tag{3.3}$$

This condition is satisfied by choosing the correct value of $u_{e\|}(r,0,t)$. To obtain a similar condition for $\phi(r,0,t)$, we kept higher order terms in $\rho_i/L_r$ in the current conservation equation to find

$$\frac{\partial}{\partial r}\left(\int_0^{L_z}\mathbf{J}_\perp \cdot \hat{\mathbf{r}}\,\mathrm{d}z\right) = 0, \tag{3.4}$$

where the average radial current is

$$\int_0^{L_z} \mathbf{J}_\perp \cdot \hat{\mathbf{r}} \, \mathrm{d}z = -\frac{\partial}{\partial t} \int_0^{L_z} \left[ \frac{n_i m_i}{B^2} \left( \frac{\partial \phi}{\partial r} + \frac{1}{e n_i} \frac{\partial p_{i\perp}}{\partial r} \right) \right] \mathrm{d}z$$
$$+ \frac{1}{B} \frac{\partial}{\partial r} \left[ \int_0^{L_z} \frac{n_i m_i}{B^2} \frac{\partial \phi}{\partial z} \left( \frac{\partial \phi}{\partial r} + \frac{1}{e n_i} \frac{\partial p_{i\perp}}{\partial r} \right) \mathrm{d}z \right]$$
$$- \int_0^{L_z} \frac{n_i m_i n_n R_{in}}{B^2} \left( \frac{\partial \phi}{\partial r} + \frac{1}{e n_i} \frac{\partial p_{i\perp}}{\partial r} \right) \mathrm{d}z$$
$$+ \int_0^{L_z} \frac{n_i m_i n_n (R_{in} + R_{\mathrm{ion}}) u_{nz}}{B} \mathrm{d}z + \int_0^{L_z} \frac{\mathbf{F}_{\mathrm{ext}} \cdot \hat{\mathbf{z}}}{B} \mathrm{d}z. \quad (3.5)$$

Equation (3.4) is the equation for $\phi(r, 0, t)$.

## 4. Model with 'separatrix'

We consider a case with mixed periodic and wall boundary condition. For $r \in [r_0, r_s[$ we apply periodic boundary conditions, whereas for $r \in [r_s, r_0 + L_r]$ we use wall boundary conditions.

The problem at $r = r_s$ is that, in general, there will be discontinuities in $\phi$ and $u_{e\|}$ because of the different treatment of these two fields on both side of $r = r_s$. These discontinuities give raise to discontinuities in the ion density, temperature and flow, and indicate that there is a boundary layer of width $\rho_i \ll L_r$ around the 'separatrix', and another boundary layer inside this layer of order $(B/B_z)\rho_e \ll \rho_i$. Here $\rho_e$ is the electron gyroradius. Within the $\rho_i$-wide layer, the potential and the ion density $n_i$, the ion and electron temperatures $T_i$ and $T_e$, the ion and electron parallel velocities $u_{i\|}$ and $u_{e\|}$, and the ion normalized distribution function $F_i$ are continuous, but the normalized electron distribution function $F_e$ will in general be discontinuous at $r = r_s$. The electron distribution function $F_e$ is continuous only within the layer of width $(B/B_z)\rho_e$. We would need to solve these layers in details to be able to connect the two sides of the 'separatrix' $r_s$.

Thankfully, these layers give straightforward results in the collisionless electron limit presented in section 5.4.2 of report 2047357-TN-05-02. In the 2D model, this limit corresponds to collisions with collision frequencies $\nu_{ss'}$ such that

$$\frac{B}{B_z} \frac{\nu_{ss'} L_z}{v_{ts}} \ll 1. \quad (4.1)$$

The advantage of this limit is that the collisional friction forces between electrons, ions and neutrals in the electron momentum equation (3.2) are small by $\sqrt{m_e/m_i} \ll 1$. Thus, even though $u_{e\|}$ is discontinuous across the 'separatrix' $r_s$, the potential $\phi$ is not affected by this discontinuity to lowest order. Indeed, without collisional friction forces, equation (3.2) simplifies to

$$0 \simeq -\frac{B_z}{B} \frac{\partial p_{e\|}}{\partial z} + \frac{e n_e B_z}{B} \frac{\partial \phi}{\partial z}, \quad (4.2)$$

and $p_{e\|}$ can be made to be continuous across $r_s$, as we proceed to argue.

Despite the fact that the electron-ion and electron-neutral collisional friction is negligible in the electron momentum equation, collisions remain important in the electron kinetic equation. As explained section 5.4.2 of report 2047357-TN-05-02, due to collisions, the electron distribution function is close to a Maxwell-Boltzmann distribution whose density and temperature are determined by quasineutrality and the electron energy equation. If $p_{e\|}$ is continuous across $r_s$ and hence $\phi$ is continuous, the boundary layer has almost no effect on ions, and the ion distribution function is continuous across $r_s$. Since the ion density determines the density of the Maxwell-Boltzmann electron distribution function via quasineutrality, it is self-consistent to assume continuity of this density. The electron temperature requires more thought. Due to the assumption $\sqrt{m_e/m_i} \ll 1$, our electron energy equation does not contain any radial transport of electron energy in the limit (4.1). Thus, it is possible to have a discontinuity (or several) in the electron temperature $T_e$, irrespective of the presence of the 'separatrix'. Whether such discontinuities exist depends on our choice of source of energy. The source can represent turbulence, for example, and turbulent diffusion would smear out discontinuities in the temperature, so we are allowed to assume that the source ensures continuity of $T_e$.

Then, if we use the simplified electron momentum equation (4.2) and we use an electron energy source that ensures that $T_e$ is continuous, we can solve the problem with the 'separatrix' by imposing

- continuity in the ion and neutral density $n_i$ and $n_n$, in the ion, neutral and electron temperature $T_i$, $T_n$ and $T_e$, and in the ion and neutral average velocities $u_{i\|}$ and $\mathbf{u}_n$;
- continuity in the ion and neutral normalized distribution functions $F_i$ and $F_n$;
- continuity in $\phi$.

The electron parallel flow $u_{e\|}$ and the electron normalized distribution function $F_e$ will be discontinuous in general.

To be able to include the friction forces in the electron momentum equation, we need to solve the layer of width $\rho_i$, and this is in general a complex gyrokinetic problem that requires a treatment similar to the one developed by Geraldini *et al.* (2017). The treatment of this layer is outside of the scope of the contract.

## 5. Discussion

We have proposed a 2D drift kinetic model for a helical magnetic field with a 'separatrix'. This model is only valid in the limit where collisions satisfy the ordering (4.1). This limit is valid for most tokamaks near the separatrix, but it is not appropriate for the far scrape-off layer, near the wall. The problem of how to connect to that far scrape-off layer in a rigorous manner thus remains open.

### REFERENCES

Geraldini, A., Parra, F.I. & Militello, F. 2017 Gyrokinetic treatment of a grazing angle magnetic presheath. *Plasma Phys. Control. Fusion* **59**, 025015.