Tests for a 2D drift kinetic model for the plasma edge

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1. Introduction

In this report we briefly discuss a strategy for testing the 2D (in configuration space) model described in detail in our previous report [1]. As the system of equations we must solve becomes more complex, analytical solutions are more difficult to obtain and are of limited utility. To address this challenge we propose to use the method of manufactured solutions [2, 3]. The basic idea behind this approach is to specify the form for the solution to the equations that we wish the code to produce, and then to derive an appropriate source term to add to the equations to make this solution consistent. In doing so, the key is to choose solutions that are closed-form, sufficiently smooth and differentiable and that are able to test a variety of terms appearing in the equations.

2. Model equations

As an illustration of the proposed method, we consider the drift kinetic system of [1] in the collisionless limit and with an assumed Boltzmann response for the electrons:

$$\frac{\partial f_i}{\partial t} - \frac{1}{B}\frac{\partial \phi}{\partial z}\frac{\partial f_i}{\partial r} + \left(\frac{v_\parallel B_z}{B} + \frac{1}{B}\frac{\partial \phi}{\partial r}\right)\frac{\partial f_i}{\partial z} - \frac{eB_z}{m_i B}\frac{\partial \phi}{\partial z}\frac{\partial f_i}{\partial v_\parallel} = S_i, \tag{1}$$

$$\frac{\partial f_n}{\partial t} + v_r\frac{\partial f_n}{\partial r} + v_z\frac{\partial f_n}{\partial z} = S_n, \tag{2}$$

and

$$n_i = n_e = N_e \exp\left(\frac{e\phi}{T_e}\right), \tag{3}$$

where f_s is the particle distribution function for species s, B is the magnetic field strength, B_z is its component along the vertical (z) direction, ϕ is the electrostatic potential, r is the radial cylindrical coordinate, t is time, $v_\parallel$ is the component of the velocity along the magnetic field, v_r and v_z are the velocity components along r and z, e is the proton charge, m_i is the ion mass, n_s is the density of species s, N_e and T_e are constants with the dimensions of density and temperature, respectively, and S_i and S_n are source terms to be determined.

Finally, the particle distribution function f_s is related to the particle density n_s via

$$n_s = \int d^3\mathbf{v} f_s. \tag{4}$$

3. Manufactured solution

We seek steady-state solutions of the form

$$\hat{f}_s = n_s \left(\frac{m_s}{2\pi T_s}\right)^{3/2} \exp\left(-\frac{m_s v^2}{2T_s}\right), \tag{5}$$

with

$$n_s(z,r) = \hat{n}_s(z) \exp\left(-\frac{r}{L_{n_s}}\right), \tag{6}$$

$$T_s(z,r) = \hat{T}_s(z) \exp\left(-\frac{r}{L_{T_s}}\right), \tag{7}$$

$$\frac{1}{L_{n_s}} = -\frac{\partial \ln n_s}{\partial r} = \text{constant}, \tag{8}$$

and

$$\frac{1}{L_{T_s}} = -\frac{\partial \ln T_s}{\partial r} = \text{constant}. \tag{9}$$

These choices for the forms of the radial density and temperature profiles are consistent with the equilibrium plasma profiles of the 'local' approximation typically employed in the closed-field-line region of tokamaks. Consequently, they are expected to be physically sensible.

With these assumptions, the ion and neutral drift kinetic equations become

$$\begin{aligned} S_i = \hat{f}_i \Bigg(& \frac{T_e}{eBL_{T_i}} \left(\frac{m_i v^2}{2T_i} - \frac{3}{2}\right) \left(\frac{\partial \ln \hat{n}_i}{\partial z} - \frac{L_{T_i}}{L_{n_i}} \frac{\partial \ln \hat{T}_i}{\partial z}\right) \\ & + v_\parallel \frac{B_z}{B} \left(\frac{\partial \ln \hat{n}_i}{\partial z} \left(\frac{T_e}{T_i} + 1\right) + \frac{\partial \ln \hat{T}_i}{\partial z} \left(\frac{m_i v^2}{2T_i} - \frac{3}{2}\right)\right)\Bigg) \end{aligned} \tag{10}$$

and

$$S_n = \hat{f}_n \left(-v_r \left(\frac{1}{L_{n_n}} + \frac{1}{L_{T_n}} \left(\frac{m_n v^2}{2T_n} - \frac{3}{2}\right)\right) + v_z \left(\frac{\partial \ln \hat{n}_n}{\partial z} + \frac{\partial \ln \hat{T}_n}{\partial z} \left(\frac{m_n v^2}{2T_n} - \frac{3}{2}\right)\right)\right). \tag{11}$$

All that is left to have closed-form expressions for S_i and S_n are specifications of $\hat{n}_s(z)$ and $\hat{T}_s(z)$. We choose $\hat{n}_s(z) = \overline{n}_s + \delta n_s \cos k_z z$ and $\hat{T}_s(z) = \overline{T}_s + \delta T_s \cos k_z z$, with $\overline{n}_s$, $\overline{T}_s$, δn_s and δT_s independent of z.

4. Future work

With manufactured solutions of the type presented in this report, we are in a position to test our numerical implementation of the 2D model presented in [1]. Furthermore, it should be straightforward to include more complicated forms for $\hat{n}_s$ and $\hat{T}_s$ in the code framework, as the Julia programming language in which the code is written supports symbolic manipulation packages that would allow for automation of this process.

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